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Willemien Kets

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Thomas Stieltjes Institute
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To my grandmothers,
E. M. Kets-de Kruijff and M. Vree-Wildbergh.

Qui tractaverunt Scientias, aut Empirici, aut Dogmatici fuerunt. Empirici, formicæ more, congerunt tantum, & utuntur: Rationales, aranearum more, telas ex se conficiunt: Apis vero ratio media est, quæ materiam ex floribus horti & agri elicit; sed tamen eam propria facultate vertit & digerit. Neque absimile Philosophiæ verum opificium est; quod nec Mentis viribus tantum aut præcipue nititur, neque ex Historia Naturali & Mechanicis Experimentis præbitam materiam, in Memoria integram, sed in Intellectu mutatam & subactam, reponit. Itaque ex harum facultatum (Experimentalis scilicet, & Rationalis) arctiore & sanctiore foedere (quod adhuc factum non est) bene sperandum est.

Those who have handled sciences have either been men of experiment or men of dogmas. The men of experiment are like the ant; they only collect and use: the reasoners resemble spiders, who make cobwebs out of their own substance. But the bee takes a middle course; it gathers its material from the flowers of the garden and field, but transforms and digests it by a power of its own. Not unlike this is the true business of philosophy; for it neither relies solely or chiefly on the powers of the mind, nor does it take the matter which it gathers from natural history and mechanical experiments and lay up in the memory whole, as it finds it, but lays it up in the understanding altered and digested. Therefore, from a closer and purer league between these two faculties, the experimental and the rational (such as has never yet been made), much may be hoped.

Francis Bacon, *Novum organum scientiarum*. Lugduni Batavorum, apud Adrianum Wijngaerde et Franciscum Moiardum (1645); English translation from *The Works of Francis Bacon*, collected and edited by J. Spedding, R. L. Ellis, D. D. Heath. Boston, Taggard and Thomson (1863).

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[1] Rumor has it that German publisher was so annoyed by the abundance of typos in books that he spend a fortune on producing "das einzige Buch ohne Druckfehler", only to find that on the title page, which was to proudly announce this fact, the "u" in the last word had been replaced by an "e".

[2] Believe me, without your imminent contempt, this thesis would have contained even more footnotes.

intellect, and creativity, however, are reflected in his work and will thus remain with us. Matt Jackson is an example to me in the way he identifies important conceptual questions. Matt, I am very happy that you have agreed to be one of my postdoc liaisons at the Santa Fe Institute. Many thanks also to you and Sara for your care and support when I fell ill during a visit to Stanford. Eric van Damme has provided constructive comments at various points. Eric, thank you also for encouragement and for help with my job search.

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Basic terms and notation

Below one finds a list of common terms and notation used throughout this thesis. The symbol ▶ refers to the section in which more information can be found on the different items.

General

$\mathbb{N}$	The set of positive integers $\{1, 2, \ldots\}$.
$\mathbb{N}_0$	The set of nonnegative integers $\{0, 1, 2, \ldots\}$.
$\mathbb{R}$	The set of real numbers $(-\infty, \infty)$.
$\mathbb{C}$	The set of complex numbers, i.e., all numbers of the form $a+bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.
Re	The real part of a complex number. E.g., if $z = a + bi$, then $\mathrm{Re}[z] = a$.
e	Euler's number, i.e., e is the unique real number such that the value of the derivative of $f(x) = e^x$ at $x = 0$ is exactly 1.
log	The natural logarithm, i.e., the logarithm to the base e.
$\log_b$	The logarithm to the base b, where $b > 0, b \neq 1$.

Game theory (▶ 2.1)

Γ	The set of all finite strategic games.
φ^p	A point-valued solution concept.
φ^s	A set-valued solution concept.

Probability and measure theory (► 2.2)

Ω	Sample space.
$\mathscr{F}$	σ-algebra.
μ	Measure, probability measure.
$\mathbb{P}$	Probability measure.
$\mathbb{E}$	Expectation.
$\xrightarrow{\text{a.s.}}$	Convergence almost surely.
$\xrightarrow{\text{d}}$	Convergence in distribution.
$\xrightarrow{\text{p}}$	Convergence in probability.
$\mathbf{1}_F$	Indicator function of the event F.

Weak set inclusion is denoted by $\subseteq$, strict set inclusion by $\subset$. The number of elements in a finite set S is denoted by $|S|$. For $k \in \mathbb{N}$, the k-fold cartesian product $\times_{i=1}^{k} S$ of a set S is denoted by S^k. Let U be a subset of a set S. The complement of U (relative to S) is denoted by U^c. We take "countable" to mean "countably infinite".

In Chapter 5, we derive some asymptotic results. We use the following standard notation. Suppose f and g are two functions of a real variable x. We say that $f(x) = O(g(x))$ as $x \to \infty$ if $|f(x)| \le M|g(x)|$ for all large x and some constant M. Similarly, $f(x) = o(g(x))$ as $x \to \infty$, if for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{R}$ such that $|f(x)| < \varepsilon|g(x)|$ for all $x > n_\varepsilon$. Finally, $f(x) \sim g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. Similar definitions hold as $x \downarrow 0$ and for real sequences $(f(n))_{n\in\mathbb{N}}$, $(g(n))_{n\in\mathbb{N}}$.

1 Networks, learning and games

Summary

This thesis is concerned with two distinct topics, (*i*) game theory and networks and (*ii*) learning in games. Here, we provide a brief introduction to these two topics and discuss the questions that we address in this thesis. We conclude this chapter with an outline of the thesis.

1.1 Networks and games

In the first part of this thesis, we focus on networks and strategic interactions. Networks play a central role in economics for two reasons. Firstly, at least since the seminal work of Coleman et al. (1966) on the diffusion of technologies and Granovetter (1974) and Rees (1966) on job contact networks, it is widely recognized that networks give access to various resources, such as information, knowledge and capital. This is corroborated by a large number of subsequent empirical studies; see e.g. Conley and Udry (2005), Foster and Rosenzweig (1995) and Powell et al. (1996) on the diffusion of new technologies, and De Weerdt (2002) and Fafchamps and Lund (2003) on informal insurance networks in developing countries. Secondly, several empirical studies have pointed out that agents' behavior is primarily shaped by the behavior of those with whom he has a direct relationship, rather than on the behavior of the population at large. Indeed, Goolsbee and Klenow (2002) and Tucker (2006) find that an individual's decision to adopt a particular communication technology is primarily influenced by the adoption decisions of those with whom he interacts directly, rather than by the overall adoption level in the population. Glaeser et al. (1996) and Topa (2001) provide similar evidence in the context of crime and unemployment, respectively.

These empirical studies have stimulated a rich theoretical literature in game theory on strategic network formation and strategic interactions on networks, starting with the seminal contributions of Myerson (1977), Jackson and Wolinsky

(1996) and Bala and Goyal (2000). This literature has produced a wealth of insights on what networks are formed by self-interested, rational agents, and what the implications are for efficiency and inequality (see Jackson, 2005, for a survey). An important branch of the literature studies the interplay of strategic network formation and strategic interactions on networks (see Goyal, 2007, for a survey).

Until recently, the game-theoretic literature on networks has focused almost exclusively on settings where agents have full information on the network structure. The assumption of complete information is particularly problematic in a network context. Since players on a network only interact with a small subset of players, it is hard to believe that they would have complete information about all other players and their relations, in particular since many social and economic networks are very large and complex (Vega-Redondo, 2007), and evolve rapidly over time (e.g. Powell et al., 2005).[1] In a sense, networks are a way to model limited communication possibilities and limited interactions, which is hard to reconcile with the assumption of complete information.

Moreover, the combinatorial constraints imposed by the network structure makes that equilibria in network games with complete information are generally hard to characterize, already in the simplest games. In addition, there is often a large number of equilibria. In such complex settings, it is hard to conceive that actual players will manage to play according to an equilibrium. To account for this, one could assume that players have incomplete information on the network, or, equivalently, that they simply do not use all information they have.[2] The task of finding an equilibrium (for game-theorists and, more importantly, for players) is then much simplified because under incomplete information, players do not have to choose their strategies taking into account the full network structure, reasoning how opponents' network position will influence their behavior, but instead choose their strategies on the basis of some probability measure over networks which is the same for each player given their local information. In effect, the assumption of incomplete information allows one to average out the discrete features of the

[1] Some empirical studies in sociology and social psychology have studied individuals' information about their network. For instance, Friedkin (1983) finds that the "observational horizon" of individuals is limited in communication networks in organizations: individuals only know their local environment in the network. Krackhardt and Hanson (1993) report that informal networks in organizations are mostly unobservable to senior executives. Also, Powell et al. (1996, p.120) observe that in R&D collaborations in biotechnology, "beneath most formal ties [. . .] lies a sea of informal relations".

[2] An alternative approach to account for the complexity involved in network games is to assume that players use simple heuristics or rules of thumb to guide their behavior. See Vega-Redondo (2007) for a discussion of this approach.

network which make the equilibria in network games hard to solve for, so that the equilibrium predictions thus obtained may provide a more realistic description of actual play.

For these reasons, attention has increasingly turned to models which allow for incomplete information or some randomness, both in contexts of network formation and of strategic interactions on a fixed network. In the first part of this thesis, we focus on the latter class of models.[3] In this introduction, we briefly discuss the literature, and we highlight some open questions that are related to the work in this thesis.

The literature on network games studies settings in which players are located on a network. Each player is associated with a node or vertex in the network, with the edges or links representing the relations between them. Players play a fixed game with their neighbors in the network. They need to take the same action in each of their interactions. Hence, while interactions are bilateral, the interactions are linked through the fact that players need to take the same action in each of the bilateral games they play. In many contexts, this is a natural assumption. For instance, suppose that a player needs to decide which operating system to buy. He may want to coordinate his choice with the choices of his neighbors to facilitate the exchange of files, but it would be too costly and impractical to buy a different operating system for each of his bilateral interactions. Furthermore, it is assumed that players have incomplete information on the network structure. They have a common prior, and in addition, they have some private information on their local environment. The literature to date has focused on the case that a player's private information is the number of connections he has in the network, i.e., his *degree* (e.g. Galeotti et al., 2006; Galeotti and Vega-Redondo, 2005; Jackson and Yariv, 2007; López-Pintado, 2006; Sundararajan, 2005).

Here, we discuss two central issues in the study of network games under incomplete information, aiming to show how the chapters in this thesis relate to them. Firstly, an important issue is how the combination of strategic interactions, incomplete information and networks can provide new insights. The second issue we address is how players' beliefs should be modeled in settings where players interact strategically on a network under incomplete information on the network structure. We discuss these issues in turn.

[3] Papers in the first class include Cabrales et al. (2007), Jackson and Rogers (2007), Marsili, Vega-Redondo, and Slanina (2004), and McBride (2006). Also see Vega-Redondo (2007) for a survey.

The interplay of networks, incomplete information and strategic interactions

The first question we address is how the combination of strategic interactions, incomplete information and networks can provide new insights in economic settings. The last few years have witnessed the development of two largely complementary literatures that model interactions on networks. The first literature is concerned with strategic interactions on networks where players have full information on the network structure (see Goyal, 2007; Jackson, 2008, and references therein). Complementary to this literature, an extensive literature has developed that focuses on stochastic processes on networks (e.g. Durrett, 2006). In the models studied in the latter literature, agents interact non-strategically with their neighbors. A primary example is the study of the spread of epidemics, where agents become infected by their neighbors, and infect their neighbors with some probability.

Both these literatures, the literature on network games with complete information and the literature on stochastic processes on networks, address the question how agents interact on a network, and in particular, how the interactions change when the network structure is varied. However, the two literatures each take a different perspective. The first literature emphasizes strategic decision-making by rational players with full information, while the latter is concerned with non-strategic decision-making in situations where agents are not informed of the full network structure.

Only recently, attention has turned to models in which strategic agents interact on a network under incomplete information, thus combining the complementary perspectives of the two literatures discussed above. The early works in this field merely build on the aforementioned literature on stochastic processes on networks, adapting the models to the specifics of the social and economic phenomena of interest (see Vega-Redondo, 2007, Ch. 4 for a discussion), and introducing strategic decision making. Two basic approaches have been adopted. The first approach studies the (long-run) behavior of myopic adjustment processes (e.g. Jackson and Yariv, 2007; López-Pintado, 2006), thus remaining close in methodological terms to the literature on stochastic processes. The second approach takes an equilibrium perspective, analyzing the (symmetric) Bayesian equilibria (Jackson and Yariv, 2007; Sundararajan, 2005). In terms of results, the two approaches are similar for certain classes of games, however, as the rest points of the stochastic process correspond to the Bayesian equilibria Bayesian equilibria of the game (Jackson and Yariv, 2007).

These two approaches already present an important step to further the understanding of strategic interactions on networks. Many important classes of games lend itself to such an approach, such as coordination games and games with strategic complementarities (Vega-Redondo, 2007), and the strategic models provide important new insights that could not be obtained from the original non-strategic models. A primary example of a question that can only be studied within strategic models is how behavior reacts to changes in incentives. For instance, Jackson and Yariv (2007) study how adoption patterns vary with the returns to adopting and adoption costs. Also, the strategic approach allows for an assessment of welfare in network settings (e.g. Sundararajan, 2005).

However, to some extent, the game-theoretic models merely rationalize the behavior of the original stochastic models. While this makes it possible to do welfare analyses and to study how behavior depends on incentives, there are some limitations to this approach. Firstly, this approach is restricted to games in which payoffs only depend on the number of neighbors taking a certain action. However, in many settings, a player's payoffs depend on *which* neighbor takes a given action, even if we restrict attention to games that are anonymous in the sense that the identity of a player per se does not matter, only his structural characteristics. Consider for instance a setting in which firms compete on multiple markets. Each firm competes on a subset of possible markets. This gives rise to a network with a group structure (see Chapter 5): each market forms a group, and firms are connected in the network if and only if they compete in at least one market. In such a setting, the effect on a firm's profit will be greater if a competitor on a large number of markets invests in a general cost-reducing technology than when a firm with which it only competes on a single market makes such an investment.

Secondly, this approach fails to capture the richness in strategic behavior that stems from the interaction between network formation and network play that has been studied in the game-theoretic literature on network games (Goyal, 2007; Jackson, 2008). For instance, in both static and evolutionary settings, it has been shown that cooperation in a (finitely repeated) prisoner's dilemma crucially depends on the possibility of players to ostracize players that defect (e.g. Hauert et al., 2007; Hirshleifer and Rasmusen, 1989; Ule, 2008).

To overcome these limitations, a further integration of the game-theoretic approach and the stochastic approach seems to be needed. While much of the work to date has tried to bridge the gap between the literatures by adding strategic decision making to models from the literature on stochastic network processes, a few papers have taken this approach one step further by abstracting from particular applications and extending the approach to a broader class of games. This is the approach taken by for instance Galeotti et al. (2006), who study games with strategic complements and substitutes in a network setting. Players do not know the network structure, but they are informed of the number of connections they have. Galeotti et al. study how the degree of players determine their choices and how this affects payoffs, showing that predictions change when the degree distribution and the correlation among players' degrees are varied.

As in the literature on stochastic processes on networks, random network models play a central role in this approach. A random network model gives for each network in a given set the probability that this network is realized (see Section 2.3 for a precise definition). The two approaches differ in an important respect in the use of these models, however. In the literature on stochastic processes on networks, random network models are used to yield (with high probability) networks that are representative of real networks in terms of their macroscopic properties. For instance, the random network model we discuss in Section 2.3.2 gives with high probability a network whose empirical degree distribution matches closely a given degree distribution (in the limit of a large number of vertices). In the literature on network games, by contrast, random network models are purely a model for players' beliefs. To the extent that players believe that the network on which they are located has certain macroscopic properties, it is of course important to use random network models that have on average these properties, but it is not necessary that the random network model produces a network with these properties with high probability. In fact, as we discuss below, it is not even clear whether we need to use belief models that give a probability distribution over networks, rather than over players' local environment in the network. This means that we can abstract from specific random network models and focus on realistic models of players' beliefs.

Such an abstract setting allows us to investigate the rich interplay between incomplete information, strategic reasoning and local interactions. Since empirical evidence on individuals' beliefs on their network, or on their information about their social environment is scarce,[4] it is important to investigate the sensitivity of game-theoretic predictions to informational assumptions and to assumptions on players' beliefs.

In Chapter 3, we study the sensitivity of game-theoretic predictions to as-

[4] Evidence suggests that agents use simple heuristics to form beliefs (Janicik and Larrick, 2005), and that their perception of the network is biased (e.g. Kumbasar et al., 1994), even in an environment with strong incentives (Johnson and Orbach, 2002).

sumptions on players' beliefs in Bayesian network games. This class of games is studied extensively in the literature (Galeotti et al., 2006; Jackson and Yariv, 2007; Sundararajan, 2005). In this class of games, players interact with their direct neighbors and are informed of their degree. Chapter 3 studies which features of players' prior beliefs are important from a game-theoretic perspective in this class of games. That is, we ask under what conditions a "small" change in players' beliefs can give rise to a large difference in outcomes. To answer this question, we have to specify what we mean by similar outcomes, i.e., when two priors are close in a strategic sense. We say that two priors are close in a strategic sense if for any game in which players hold one of these priors, for any symmetric Bayesian-Nash equilibrium in that game, there is a symmetric approximate equilibrium in the associated game with the other prior such that ex ante expected payoffs are close under both equilibria. If that is the case, players can obtain approximately the same ex ante expected payoffs under both priors.

We show that two priors are close in a strategic sense if and only if the degree distribution and the correlation among neighbors' degrees is similar under the two priors. This is an important result, for two reasons. Firstly, this result implies that in order to explore the full range of strategic outcomes in this class of network games, it is sufficient to vary the degree distribution and the degree correlation. Hence, on the one hand, varying the type distribution, as has been the focus of much of the literature so far (e.g Jackson and Yariv, 2007; Sundararajan, 2005), is often not enough. On the other hand, this result limits the set of priors that one needs to consider: priors need only be varied on two dimensions. A second important implication is that we can interpret such network game as a set of overlapping "local games", and that we do not need to concern ourselves with the nonlocal features of players' beliefs. That is, while players' beliefs are represented by a probability measure on a set of networks, we show that it is only the induced degree distribution and the induced degree correlation that matters for game-theoretic outcomes.

Chapter 4 takes this approach one step further by allowing for uncertainty over the network size. This is a natural assumption in the current context. If a player only interacts with a small subset of players and has incomplete information on the network structure, it is natural to assume that he does not know the exact size of the network. Interestingly, we obtain qualitatively different results when networks can be of any finite size. When players are uncertain about the network size, two priors are strategically close if and only if (*i*) they assign similar prior probabilities to all events involving a player and his neighbors, (*ii*) with high probability, a player believes, given his type, that his neighbors' conditional beliefs are similar under the two priors, and that his neighbors believe, given their type, that. . . the conditional beliefs of their neighbors are similar, for any number of iterations.

Condition (*i*) is similar to the condition for strategic closeness in Bayesian network games derived in Chapter 3. Condition (*ii*) is new. The reason why we need an additional condition for strategic closeness in network games with uncertainty over the network size is that when a network can be of any finite size, the type set is countably infinite (recall that a player's type is his degree in these games). In that case, events that have small prior probability may considerably affect outcomes through players' conditional beliefs: even if an event has small prior probability, this event may influence a player's actions when he thinks (given his private information) that it is likely that his neighbor think it is likely that their neighbors think it is likely. . . that the small probability event is true. When the type set is finite, as in the class of Bayesian network games studied in Chapter 3, closeness in terms of prior probabilities assigned to local events implies closeness in terms of players' conditional beliefs (cf. Proposition 4.5.15). This shows that it is important to be careful in specifying the model. While it seems innocuous to allow for uncertainty over the network size, the results in Chapter 4 shows that it can have important ramifications.

Interestingly, condition (*ii*) can also be formulated in terms of correlations among types. An alternative formulation of condition (*ii*) is that the set of types for which conditional beliefs are similar has to have high probability, and has to be sufficiently cohesive in the sense that with high conditional probability, a type in that set interacts only with types in that set that, with high conditional probability, only interact with types in that set, and so on. This formulation of condition (*ii*) is reminiscent of the results obtained by Morris (2000) on contagion on networks. In the setting of Morris, players on a fixed network play a coordination game with their neighbors, with their payoffs depending on the fraction of their neighbors taking a certain action. He finds that, starting from a finite set of players X, behavior does *not* spread contagiously by myopic best-reply dynamics on a network with a countably infinite number of players if and only if the network of players not belonging to X contains a large group of players Y that is sufficiently cohesive, in the sense that players from Y interact mostly with other players from Y, who in turn interact primarily with other players from Y, and so on.

Condition (*ii*) is a direct stochastic analogue of this result of Morris (2000). Rather than a fixed network of players, we consider a random network of players, which induces a fixed interaction structure on the players' types, and the situation we consider is the following. Suppose that there is a set of types with small prior probability for whom conditional beliefs are very different under two priors (so that they may follow different strategies under the two priors). We ask under what conditions these types do not “infect” a large (in terms of ex ante probability) set of types through players’ higher order beliefs. This is the case precisely when there is a group of types with high prior probability that is sufficiently cohesive.

That is, we map a random network of *players* to a fixed interaction structure of *types*. This mapping allows us to apply the formal equivalence between games on a fixed network with complete information and Bayesian games established by Morris (1997, 2000), which in turn enables us to use ideas from literature on higher order beliefs. This provides an example of the richness of results that can be gained from more fully integrating the strategic approach with the stochastic approach. Moreover, the work presented in Chapter 4 opens the door to the application of ideas and concepts from the literature on higher order beliefs to answer questions that are important in network settings, such as the sensitivity of predictions to assumptions on players’ information.

Beliefs and random network models

The second question we address in this introduction is how players’ beliefs in network games should be modeled. So far, the literature, including the work in this thesis, has primarily used random network models to model players’ beliefs, focusing on local features of these models, such as the degree distribution induced by the model. Here, we ask whether these types of models are the most appropriate representation of players’ beliefs, and how they might be improved. We thereby focus on two issues. Firstly, we ask whether we need models that specify players’ beliefs over the full network. Secondly, we discuss whether we need to account for features of networks that are not purely local, such as community structure.

To start with the first issue, the random network models employed in much of the literature on network games present a “global” model of players’ beliefs, in the sense that they provide a probability measure on a set of networks. Hence, when we use these models to represent players’ beliefs, we assume that players have some assessment of how likely it is that each network is realized. However, it is not clear that these global models of beliefs present an adequate representation of the beliefs of human subjects or, more generally, economic agents. Random network models, especially when they involve some interdependencies among the characteristics of players (vertices) in the network, may be quite complicated. It may be more reasonable to assume that players only entertain local beliefs, i.e., beliefs about the characteristics of their neighbors. An important characteristic of network games is that players play games with their direct neighbors, who in turn interact with their neighbors, etcetera. The work in Chapter 3 and 4 indicates that it is not so much the global structure of the network that matters for game-theoretic predictions, rather, it is the induced interactions between player types that play a role, as in standard Bayesian games. This insight may provide a way to move away from global belief models to more realistic models of players' beliefs.

While it may be more realistic to assume that players do not have beliefs over the full network structure, there may be network properties that are not purely local that are important for game-theoretic applications. The second issue we address here is therefore whether "meso-level" features of networks, i.e., features that are neither purely local nor global, should be accounted for. In the games studied in the literature so far, a player's payoffs only depend on his own action and type and the actions and types of his neighbors. That is, higher-level network structures such as the partition of the network into communities (densely linked groups) do not play a role. In Chapter 3 and 4, we show that only players' beliefs over the distribution of degrees and the correlation between the degrees of neighbors affect game-theoretic predictions. However, empirical evidence indicates that also meso-level features of the network affect actions and outcomes. For instance, Coleman (1990) argues that networks which exhibit greater network closure, i.e., that are highly interconnected, generate high trust. Several empirical studies support this claim (e.g. Allcott et al., 2007), indicating that the community structure of networks plays an important role.

An important next step would therefore be to widen the scope to other classes of games, where players' payoffs may depend on other features of the network. In network games with incomplete information on the network structure in which closure or community structure plays an important role, other features of priors than the degree distribution and the degree correlation are likely to be important for game-theoretic predictions, which may yield new insights. Chapter 5 proposes a random network model with a community structure which can be used to model beliefs in such game-theoretic models, as it allows one to gradually vary features of the model such as the degree distribution and community structure.

1.2 Learning in games

Up to recently, work in noncooperative game theory has focused almost exclusively on equilibria in games, in particular on Nash equilibria and refinements of the Nash equilibrium concept (see Section 2.1 for formal definitions and discussion). When one views game theory as a positive science, i.e., when one takes the stance that the principal aim of game theory is to provide an accurate description of the behavior of players in strategic situations, an important question is when and why we can expect players to behave according to some equilibrium. Traditionally, the explanation is that equilibrium behavior results from analysis and introspection of players in situations in which the rules of the game, the rationality of the players, and the utility functions of players are common knowledge (e.g. Myerson, 1991; Osborne and Rubinstein, 1994). However, this approach suffers from a number of conceptual and empirical problems (Fudenberg and Levine, 1998).[5]

For these reasons, an alternative explanation has become increasingly popular over the last few years. Under this alternative explanation, equilibrium play may be the long-run outcome of an adaptive process in which boundedly rational players learn that some strategies perform better than others. The literature on learning and adaptive play in games (see e.g. Fudenberg and Levine, 1998; Samuelson, 1997; Weibull, 1995, and references therein) proposes various adaptive processes to describe players' behavior, and analyzes the long-run behavior of these processes. In the second part of this thesis, we investigate several issues in the theory of learning in games. Here, we discuss how the chapters of this thesis fit in with the literature on learning in games.

Before we begin our discussion, it is important to remark that learning models are *not* models that explain how players learn to play according to an equilibrium. This would presume, firstly, that players always learn to play according to an equilibrium, and, secondly, that learning models always converge to an equilibrium. Both of these presumptions are false. Firstly, a number of experimental studies report that subjects do not learn to play according to a Nash equilibrium of the game (e.g. Erev and Roth, 1998; Ochs, 1995). Secondly, learning models need not converge, or need not converge to an equilibrium. In fact, Hart and Mas-Colell (2003) show that there exists no uncoupled learning dynamic, i.e., a learning dynamic in which the changes in a player's strategy does not depend on the payoff functions of other players, that guarantees Nash convergence.[6]

5 An important conceptual problem arises for instance when there are multiple equilibria: if we cannot explain how players come to expect the same equilibrium, their play need not correspond to any equilibrium of the game. A major empirical problem with the traditional explanation of equilibrium behavior is that play in the early rounds of many experiments does not resemble equilibrium play, which is not in line with the idea that equilibrium behavior results from analysis and introspection. See Fudenberg and Levine (1998) for a discussion of these and other problems with the traditional explanation of Nash equilibrium play.

Rather than explaining how players learn to play according to an equilibrium, learning theory contributes to our understanding of play in games in four distinct ways. Firstly, learning theory may offer an accurate description of how experimental subjects learn to play according to an equilibrium when they are initially very far from equilibrium (e.g. Cooper et al., 1997). Secondly, when experimental play converges, but not to a Nash equilibrium of the game, then learning models may give an accurate account of how subjects choose their actions, and thus shed some light on why the equilibrium prediction fails (e.g. Erev and Roth, 1998). Thirdly, learning theory can help resolve equilibrium selection problems (Samuelson, 1997). Finally, learning models can suggest useful ways to evaluate and modify traditional equilibrium concepts. In some cases, learning models converge to a refinement of Nash equilibrium, such as risk-dominant equilibria (e.g. Kandori et al., 1993; Young, 1998); in other cases, predictions are much weaker than Nash equilibrium, for instance in extensive form games.

The point of departure of learning models is that players may not always be perfectly rational: players choose their strategies in a trial-and-error learning process, finding that some strategies perform better than others. At the same time, players may be somewhat sophisticated. Learning models differ in the degree of sophistication they attribute to players, aiming to strike a balance between the realization that players are not fully rational yet not completely unsophisticated.[7]

In Chapter 6, we present a model that tries to strike such a balance. In the model we propose there, players choose best replies to beliefs that are supported by observed play of their opponents in the recent past, thus requiring some sophistication on the part of players. At the same time, players are "forgetful" (cf. Hurkens, 1995; Young, 1998): they only recall a fixed number of past periods. In addition, players are subject to a so-called "recency bias": if there are multiple

[6] Convergence and stability properties of learning models generally depend on the properties of the game, so that a learning model may converge to an equilibrium in one class of games but not in another. For instance, the reinforcement learning model of Roth and Erev (1995) does not converge in games with a unique mixed-strategy equilibrium. Stochastic fictitious play (e.g. Fudenberg and Levine, 1998, Ch. 4) converges in this class of games, but not to the Nash equilibrium.

[7] For instance, reinforcement learning models (Roth and Erev, 1995) only posit that strategies that have been successful in the past, will be used more often than strategies that have proved less successful. To learn according to a reinforcement learning model, players only need to know the actions they chose in the past and the associated payoffs. In particular, they need not know or understand the structure of the game. By contrast, in belief-based models (Fudenberg and Levine, 1998), players form beliefs over their opponents' behavior on the basis of their past play. To learn according to this class of models, players need to be rational, form beliefs over others' play and calculate expected payoffs. For this, they need to be informed of the game, their opponents' past actions and their own history of play.

best-replies to a given belief, players choose the most recent one.

We show that play converges to a minimal prep set of the game. Minimal prep sets (Voorneveld, 2004) are a set-valued solution concept for strategic games (see Section 2.1.2 for a precise definition). Also in the definition of this solution concept there is a balance between relaxing the assumption of full rationality and assuming some sophistication. The solution concept combines a standard rationality condition, stating that the set of recommended strategies to each player must contain at least one best reply to whatever belief he may have that is consistent with the recommendations to the other players, with players' aim at simplicity, which encourages them to maintain a set of strategies that is as small as possible. The work presented in Chapter 6 thus provides an instance of the fourth potential contribution of learning theory listed above, by providing a dynamic motivation for minimal prep sets.

A recurrent theme in the theory of learning in games is that different learning processes make different predictions (for a given class of games). It is hard to understand in general terms why some learning models converge to one solution concept, and other learning models to another. To further our understanding, it is important to understand the nature of different solution concepts. In Chapter 7 we therefore provide an axiomatic characterization of minimal prep sets and the closely related minimal curb sets (Basu and Weibull, 1991). Minimal curb sets are a set-valued solution concept for strategic games that requires the set of recommendations to players to contain not just some (as in minimal prep sets), but *all* best responses against beliefs restricted to the recommendations to the remaining players (see Section 2.1.2 for a precise definition).

We show that for the most part, minimal curb sets and minimal prep sets satisfy the same desirable properties; in their respective axiomatizations, the only distinguishing property concerns the treatment of one-player games. This follows directly from the definitions: the concept of a minimal prep set requires players to hold some best reply to any belief they may have that is consistent with the recommendations to other players, while the concept of minimal curb sets requires them to hold all best responses. The interesting issue is that while the two solution concepts sometimes give rise to very different solutions (see Tercieux and Voorneveld (2005) for some appealing examples), all differences between the two solution concepts are captured by the way they deal with one-player games.

In Chapter 8 and 9, we turn to the study of learning in a class of congestion games, the class of minority games. In a minority game, players need to choose between two actions; the players who have chosen the action that is selected by the smallest number of players receive the highest payoff. In Chapter 8, we characterize the limiting behavior of several well-known learning models for this class of games. Interestingly, we find that predictions are not equivocal. While this may be partly due to the fact that the games in the class we study have a continuum of equilibria (cf. Duffy and Hopkins, 2005), it is nevertheless an interesting result, as some important related games share this property. A primary example is the class of market entry games (Selten and Güth, 1982) when the market capacity is integer (Duffy and Hopkins, 2005).[8] These games have been studied extensively experimentally, with puzzling results (see Ochs, 1999, for a survey). While aggregate play is largely consistent with equilibrium play, a "magic" finding, in the words of Kahneman (1988), with the number of entrants close to capacity, individual play generally does not resemble Nash play.

The work reported in Chapter 8 may help shed some light on this. While the set of equilibria of the class of market entry games with integer capacity (Duffy and Hopkins, 2005) is very similar to the class of equilibria in minority games, the two classes of games differ in one important respect: the symmetry of the minority game makes it harder for players to play repeated-game strategies in experiments. In experiments on the (asymmetric) market entry games, players can try to build a reputation for entering (see Duffy and Hopkins, 2005, for a discussion). There is no scope for such strategies in minority games. By comparing experimental findings in minority games and market entry games, one could gain further insight in the question why players do not learn to play according to an equilibrium in the market entry game. By characterizing the long-run behavior of different learning processes in the minority game, Chapter 8 provides a useful benchmark to assess players' behavior in experiments, and thus aids in our understanding of why players do not learn to play according to an equilibrium in certain games, in line with the second potential contribution of learning theory listed above.

In Chapter 9, we examine an alternative learning model that may provide a realistic model for players' behavior in congestion games. Experimental findings on such games seem hard to explain using standard learning models, such as the ones discussed in Chapter 8. Chapter 9 therefore discusses an alternative learning model. In this model, players condition their behavior on a limited history of past outcomes, using so-called response modes that prescribe which action to take given the history of recent outcomes. Players decide which response mode to use

[8] In a market entry game, players need to decide whether to enter a market or not. Payoffs to entering generally fall in the number of entrants, while not entering gives a payoff independent of the actions of other players.

on the basis of the past performance of different response modes. We show that such a learning model may provide a good description of experimental play in such games, in line with the second potential contribution of learning theory listed above.

1.3 Outline of this thesis

We conclude this introduction with a brief outline of this thesis. Chapter 2 provides the necessary theoretical background. It introduces the three main mathematical fields on which the chapters in this thesis build: game theory, probability and measure theory, and the theory of random networks. Each of the following chapters is meant to be self-contained, although it is assumed that the reader is familiar with the concepts and results presented in Chapter 2. Whenever necessary, the reader is referred to the definitions and results of Chapter 2.

Part I of this thesis concentrates on networks and game theory. Chapter 3 studies a setting in which players are located on a network and have incomplete information on the network structure. In these Bayesian network games, players' beliefs are represented by a (common) prior over a set of networks. Each player is informed of the number of neighbors he has in the network, i.e., a player's type is his degree. The chapter studies the sensitivity of game-theoretic predictions to the specification of players' beliefs in such a setting. We show that a necessary and sufficient condition for two priors to be close in a strategic sense is that they be similar in terms of the prior probabilities assigned to "local" events, i.e., events involving the types of a player and his neighbors.

In Chapter 4, we study a similar question as in Chapter 3. In contrast with Chapter 3, we now allow for uncertainty over the network size. We show that in this case, small probability events can have an important effect on outcomes through players' conditional beliefs: a player may think it is likely (given his type) that his neighbors think it is likely that... the small probability event is true. In addition to requiring that two priors be similar in terms of the prior probabilities assigned to local events for them to be close in a strategic sense, we also need that with high probability, a player has a type such that his conditional beliefs are close and that he believes (given his type) that with high probability, his neighbors' conditional beliefs are close, and that they believe (given their type), that their neighbors' conditional beliefs are close, etcetera.

The final chapter of Part I, Chapter 5, proposes a random network model with a group structure that can be used to model players' beliefs in network games. We characterize the degree distribution and the clustering of the model, showing that we can obtain a random network with any clustering and any degree distribution. Such a model allows for the simultaneous investigation of the effect of players' beliefs on the degree distribution on the clustering on game-theoretic outcomes. Moreover, the model is set up in such a way that it could be the result from a strategic process of network formation.

Part II of this thesis is concerned with learning in games. Chapter 6 studies a best-reply learning process in which there is some inertia in players' behavior. More precisely, we study a best-reply learning process in which players form beliefs over others' actions on the basis of recent past play. In addition, we assume that players players have a recency bias in the sense that they always choose the most recent best reply to a given belief whenever there is more than one best reply. We show that play converges to minimal prep sets, a set-valued solution concept that combines a standard rationality condition, stating that the set of recommended strategies to each player must contain at least one best reply to whatever belief he may have that is consistent with the recommendations to the other players, with players' aim at simplicity, which encourages them to maintain a set of strategies that is as small as possible. We thus provide a dynamic motivation for this solution concept.

In Chapter 7, we give an axiomatic characterization of minimal prep sets and the related solution concept of minimal curb sets. We show that both solution concepts satisfy the axiom of consistency: given that a set of players commits to playing according to a certain solution, the remaining players in the reduced game should have no incentive to deviate from it. While this seems to be a minimal requirement for solution concepts, it can be shown that refinements of the Nash equilibrium are not consistent (when one requires utility maximizing behavior in one-player games and nonemptiness). Hence, minimal prep sets and minimal curb sets are attractive concepts in this respect.

We then turn to learning in a specific class of congestion games. In Chapter 8, we study the limiting behavior of several well-known learning processes in this class of games, and we show that different learning models yield different predictions. This is an important finding, since experimental results on such games are puzzling in the sense that whereas play quickly converges to equilibrium play on the aggregate level, individual behavior does not conform to a Nash equilibrium. That different learning models give different predictions in this class of games means that we have little guidance from learning theory to explain these experimental findings. In Chapter 9, we therefore discuss an alternative learning model for this class of games, and relate it to other learning models in game theory. The model we discuss assumes that each player is endowed with a random set of so-called response modes that specify a player's response to a given history of play. Each player chooses the response mode from his endowment that has performed well in the past. We argue that this is a behaviorally plausible model for such games, as it allows players to coordinate to differentiate in a natural way. Moreover, it can explain the experimental findings discussed above, since the model predicts aggregate equilibrium play, while individuals do not play according to a Nash equilibrium.

2 Preliminaries

Summary

In this chapter, we briefly introduce the three main mathematical fields from which we draw in this thesis: game theory, probability and measure theory, and the theory of random networks. In addition, we list some basic results from these fields. Section 2.1 provides a brief introduction to game theory. Section 2.2 gives an overview of the concepts and results we use from probability and measure theory. Finally, Section 2.3 gives an introduction to the field of random networks.

2.1 Game theory

In this section, we introduce some basic game-theoretical notions. We do not intend to cover the whole field; rather, we focus on the concepts and definitions that are important for the work in this thesis. In particular, we restrict attention to noncooperative game theory, i.e., we assume that players cannot make binding agreements. Furthermore, we focus on one-shot strategic games and Bayesian games. Good textbook treatments of game theory include Fudenberg and Tirole (1991), Myerson (1991), and Osborne and Rubinstein (1994).

This section is organized as follows. In Section 2.1.1, we introduce the class of strategic games. We discuss several solution concepts for this class of games in Section 2.1.2. In Section 2.1.3, we discuss the class of Bayesian games.

2.1.1 Strategic games

A *(finite strategic) game* is a tuple $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where

- N is a nonempty, finite set of *players*;
- for each player $i \in N$, A_i is a nonempty, finite set of pure strategies or *actions*;
- for each player $i \in N$, u_i is a *von Neumann-Morgenstern utility function* from the set of strategy profiles $\times_{i \in N} A_i$ to $\mathbb{R}$, which specifies for each strategy profile $a = (a_j)_{j \in N} \in \times_{i \in N} A_i$ the *payoff* of a to i.

		Steve	
		Pane	Pear
Bill	Pane	10, 5	0, 0
	Pear	0, 0	5, 10

Figure 2.1. The payoff matrix of the game of Example 2.1.1

Let $n := |N|$. We denote the set of all finite strategic games by Γ.

When the number of players is equal to two, a game can be conveniently represented by means of its *payoff matrix*, which gives the payoffs to each player for each choice of action profiles. Example 2.1.1 presents a simple game and its payoff matrix.

Example 2.1.1. Bill and Steve need to decide which operating system to buy. They can choose between two options, Pear and Pane. Both have their preferences over the two systems, but in addition, each cares about the other's choice, as they would like to exchange files with each other. More specifically, if they buy a different operating system, they each receive a payoff of 0. If they both buy the Pane system, Bill gets a payoff of 10, and Steve gets a payoff of 5, while if they both buy the Pear system, Bill gets a payoff of 5, and Steve gets a payoff of 10. The payoff matrix of this game is given in Figure 2.1. $\triangleleft$

We write $A = \times_{i \in N} A_i$. Let $i \in N$. We use the standard notation A_{-i} to denote the set of strategy profiles $\times_{j \in N \setminus \{i\}} A_j$ of the opponents of i, with typical element $a_{-i} = (a_j)_{j \in N \setminus \{i\}}$. Let $S \subseteq N$. Then, $A_S = \times_{j \in S} A_j$ denotes the set of strategy profiles of players in S, with typical element a_S. With slight abuse of notation, we sometimes represent a strategy profile $a = (a_j)_{j \in N} \in A$ by (a_i, a_{-i}) or $(a_S, a_{N \setminus S})$ to stress the action choice of player i or players in S.

Players are allowed to use *mixed strategies*. A player who uses a mixed strategy, randomizes over (a subset of) his actions. Let Q_i be a nonempty subset of A_i. The set of mixed strategies of player $i \in N$ with support in Q_i is denoted by $\Delta(Q_i)$, i.e.,

$$\Delta(Q_i) := \Big\{\alpha_i : Q_i \to [0,1] \,\Big|\, \sum_{q_i \in Q_i} \alpha_i(q_i) = 1\Big\}.$$

The set of all mixed strategies of player $i \in N$ is thus $\Delta(A_i)$. Given a mixed strategy profile $\alpha = (\alpha_j)_{j \in N} \in \times_{j \in N} \Delta(A_j)$, we use α_{-i} to denote $(\alpha_j)_{j \in N \setminus \{i\}}$.

We can extend payoffs to mixed strategies. The payoffs to player $i \in N$ of a mixed strategy profile $\alpha = (\alpha_j)_{j \in N}$ are given by:

$$u_i(\alpha) = \sum_{a \in A} \prod_{j \in N} \alpha_j(a_j)\, u_i(a).$$

That is, the payoff of a mixed strategy profile α is simply the expected payoff associated with the lottery α. Similarly, the payoffs to player i of action $a_i \in A_i$ when the other players play according to the mixed strategy profile $\alpha_{-i} = (\alpha_j)_{j \in N \setminus \{i\}}$ are denoted by

$$u_i(a_i, \alpha_{-i}) = \sum_{a_{-i} \in A_{-i}} \prod_{j \in N \setminus \{i\}} \alpha_j(a_j)\, u_i(a_i, a_{-i}).$$

Given a mixed strategy profile $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$, an action $a_i \in A_i$ is a (pure) best response or *(pure) best reply* of i to profile α_{-i} if the payoff of a_i given that other players play according to α_{-i} is at least as high as the payoff of any other action $b_i \in A_i$. The set of pure best replies of player i against profile α_{-i} is denoted by:

$$BR_i(\alpha_{-i}) = \{a_i \in A_i \mid \forall b_i \in A_i : u_i(a_i, \alpha_{-i}) \geq u_i(b_i, \alpha_{-i})\}.$$

Each player forms beliefs over the strategies of his opponents. We take beliefs to be profiles of mixed strategies, i.e., correlation in beliefs is not allowed. We can therefore identify the set of pure best responses against the *belief* $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$ of player i with the set $BR_i(\alpha_{-i})$ of pure best responses against the *mixed strategy profile* α_{-i}.

2.1.2 Solution concepts for strategic games

In Part II of this thesis, we study different solution concepts. A solution concept for strategic games provides for each (finite strategic) game a prediction of play. In this thesis, we consider both point-valued and set-valued solution concepts for strategic games. A point-valued solution concept assigns to each game a collection of strategy profiles. Formally:

Definition 2.1.2. *A* point-valued solution concept φ^p *on* Γ *is a correspondence that assigns to each game* $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ *a collection* $\varphi^p(G)$ *of mixed strategy profiles, i.e., each element of* $\varphi^p(G)$ *(if there is one) is a mixed strategy profile* $\alpha \in \times_{i \in N} \Delta(A_i)$*. We call elements* $\alpha \in \varphi^p(G)$ solutions *of* G*.*

By contrast, set-valued solution concepts assign to each game a collection of product sets of (pure) strategies:

Definition 2.1.3. *A* set-valued solution concept φ^s *on* Γ *is a correspondence that assigns to each game* $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$ *a collection* $\varphi^s(G)$ *of product sets in* A*, i.e., each element of* $\varphi^s(G)$ *(if there is one) is a set* $Q = \times_{i\in N} Q_i$ *with* $Q_i \subseteq A_i$ *for each* $i \in N$*. We call elements* $Q \in \varphi^s(G)$ solutions *of* G*.*

We discuss these classes of solution concepts in turn. Point-valued solution concepts are the standard class of solution concepts studied in noncooperative game theory. The most commonly studied solution concept is the Nash equilibrium concept.

Definition 2.1.4. *A pure strategy profile* $a \in A$ *is a* pure Nash equilibrium *of a game* $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ *if no player can benefit from a unilateral deviation:*

$$\forall i \in N, \forall b_i \in A_i : \quad u_i(a) \geq u_i(b_i, a_{-i}).$$

A pure Nash equilibrium is strict *if the above inequality is strict whenever* $b_i \neq a_i$*. Similarly, a mixed strategy profile* $\alpha \in \times_{i\in N}\Delta(A_i)$ *is a* mixed strategy Nash equilibrium*, or simply a* Nash equilibrium*, if*

$$\forall i \in N, \forall b_i \in A_i : \quad u_i(\alpha) \geq u_i(b_i, \alpha_{-i}).$$

Using Kakutani's fixed point theorem (Ok, 2007), it is straightforward to show existence of Nash equilibria (possibly in mixed strategies) for finite strategic games. Various refinements of the Nash equilibrium concept, in addition to strict equilibria, are discussed in Van Damme (1991).

We now turn to set-valued solution concepts. Set-valued solution concepts form a natural class of solution concepts for a number of reasons. Firstly, in many contexts, people live by rules and principles that restrict behavior but do not determine it uniquely. Hence, it may be more natural to think of people choosing freely among a subset of possible actions rather than choosing a fixed distribution over this set of actions, as in a mixed strategy Nash equilibrium. Secondly, many learning processes settle down in sets, rather than converging to strategy profiles (e.g. Hurkens, 1995; Tercieux, 2006; Young, 1998, and Chapter 6 of this thesis). Thirdly, set-valued solutions are the natural outcome of a process of iterated elimination of "bad" actions. For instance, the set of rationalizable strategies of a game is the set of strategies that survive the iterated removal of strategies that are never a best response. Finally, as discussed in Chapter 7 of this thesis, set-valued solution concepts satisfy the desirable property of consistency (Peleg et al., 1996; Peleg and Tijs, 1996), while recommending strategy profiles to players can lead to consistency problems (Norde et al. (1996); see Voorneveld et al. (2005) for a discussion). Set-valued solution concepts include the set of rationalizable strategies (Bernheim, 1984), persistent retracts (Kalai and Samet, 1984), minimal curb sets (Basu and Weibull, 1991), and minimal prep sets (Voorneveld, 2004). Here, we focus on minimal curb sets and minimal prep sets, as these concepts play an important role in Chapter 6 and 7 of this thesis.

Basu and Weibull (1991) define the concepts of curb sets and minimal curb sets, where "curb" is mnemonic for "closed under rational behavior".

Definition 2.1.5. *A* curb set *of a game* $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$ *is a nonempty product set* $Q = \times_{i\in N} Q_i \subseteq A$ *such that for each* $i \in N$ *and each belief* $\alpha_{-i} \in \times_{j\in N\setminus\{i\}}\Delta(Q_j)$ *of player* i*, the set* Q_i *contains* all *best replies of player* i *against his belief:*

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(Q_j) : \quad BR_i(\alpha_{-i}) \subseteq Q_i.$$

A curb set Q *is* minimal *if no curb set is a proper subset of* Q*.*

The set-valued solution concept that assigns to each game its collection of minimal curb sets is denoted by min-curb. Hence, for each game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$:

$$\text{min-curb}(G) = \{Q \subseteq A \mid Q \text{ is a minimal curb set of } G\}.$$

Similarly,

$$\text{curb}(G) = \{Q \subseteq A \mid Q \text{ is a curb set of } G\}.$$

Voorneveld (2004) introduces the concept of prep sets and minimal prep sets, where "prep" is shorthand for "preparation".

Definition 2.1.6. *A* prep set *of a game* $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$ *is a nonempty product set* $Q = \times_{i\in N} Q_i \subseteq A$ *such that for each* $i \in N$ *and each belief* $\alpha_{-i} \in \times_{j\in N\setminus\{i\}}\Delta(Q_j)$ *of player* i*, the set* Q_i *contains* at least one *best reply of player* i *against his belief:*

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(Q_j) : \quad BR_i(\alpha_{-i}) \cap Q_i \neq \emptyset.$$

A prep set Q *is* minimal *if no prep set is a proper subset of* Q*.*

The set-valued solution concept that assigns to each game its collection of minimal prep sets is denoted by min-prep. Hence, for each game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$:

$$\text{min-prep}(G) = \{Q \subseteq A \mid Q \text{ is a minimal prep set of } G\}.$$

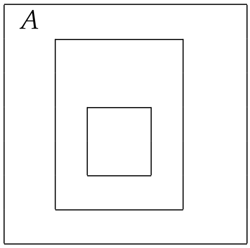


Figure 2.2. Each finite strategic game has a minimal curb set (minimal prep set).

Similarly,

$$\text{prep}(G) = \{Q \subseteq A \mid Q \text{ is a prep set of } G\}.$$

Establishing existence of minimal curb sets and minimal prep sets in finite strategic games is simple. The entire pure-strategy space A is a curb set (prep set). Hence the collection of curb sets (prep sets) is nonempty, finite (since A is finite) and partially ordered by set inclusion (see Figure 2.2 for an illustration of this argument). This proves:

Proposition 2.1.7. *Each game $G \in \Gamma$ has at least one minimal curb set and at least one minimal prep set, i.e.,*

$$\text{min-curb}(G) \neq \emptyset, \qquad \text{min-prep}(G) \neq \emptyset.$$

See Basu and Weibull (1991, Prop. 1) and Voorneveld (2004, Thm. 3.2) for general existence results for minimal curb sets and minimal prep sets, respectively.

Example 2.1.8 illustrates the differences between the different solution concepts discussed here.

Example 2.1.8. In the two-player game G in Figure 2.3, min-curb$(G) = \{\{T, B\} \times \{L, R\}\}$, min-prep$(G) = \{\{T\} \times \{L\}\}$, and the set of Nash equilibria consists of all mixed strategy profiles $(\alpha T + (1 - \alpha)B, L)$ with $\alpha \in [1/2, 1]$. The pure Nash equilibrium (T, L) can be obtained by iterated elimination of weakly dominated actions; in this example, this is exactly the outcome predicted by the game's unique minimal prep set $\{T\} \times \{L\}$. ◁

	L	R
T	1,1	1,0
B	1,0	0,1

Figure 2.3. Differences between min-curb, min-prep, and Nash equilibria.

2.1.3 Bayesian games

In many cases, players are uncertain about the characteristics of some or all of the other players. For instance, they may be uncertain about the payoffs other players receive from their actions. Such a situation is modeled concisely by means of Bayesian games.

More precisely, let $n \in \mathbb{N}$, and let N be a set of n players. Each player $i \in N$ is endowed with some nonempty, finite set of *actions* A_i. Uncertainty is modeled by introducing a set of possible *states of nature* Ω, each of which is a description of all relevant characteristics of the players in N. Throughout this thesis, we assume that the set of states of nature is finite. Players have a *(common) prior* μ over the set of states of nature, i.e., a probability measure that for each state of nature gives the probability that players assign to the state before they have any information on the state.[1] In any given play of the game, some state $\omega \in \Omega$ is realized. Players' information about the state of nature is modeled by a profile of *signal functions* $(\tau_i)_{i \in N}$. For each $i \in N$, τ_i is a function on Ω that gives for each state $\omega \in \Omega$ the signal $\tau_i(\omega)$ that player i observes when the state of nature is ω. For $i \in N$, denote the set of all possible signals for player i by T_i. By finiteness of Ω, the set T_i is finite. We refer to $\tau_i(\omega)$ as the *type* of player i in state ω, and to T_i as the *type set* of i. After players have learned their signals, they update their beliefs according to Bayes' rule. More specifically, when the common prior is μ, the posterior or *conditional belief* of player $i \in N$ of type $t_i \in T_i$ that the state is $\omega \in \Omega$, is given by

$$\frac{\mu(\{\omega\} \cap \tau_i^{-1}(t_i))}{\mu(\tau_i^{-1}(t_i))} =: \mu\Big(\{\omega\}\Big|\tau_i^{-1}(t_i)\Big)$$

when $\tau_i^{-1}(t_i) \subseteq \Omega$ has positive probability under μ. Players' payoffs depend on their action and the actions of other players as well as on the state. More precisely, for each $i \in N$, the payoffs to player i are given by a *von Neumann-Morgenstern*

[1] For a formal definition of probability measures, see Section 2.2.1.

utility function $u_i : \times_{j \in N} A_j \times \Omega \to \mathbb{R}$, with for each $a \in \times_{j \in N} A_j$ and $\omega \in \Omega$, $u_i(a, \omega)$ the payoffs to action profile a when the state is ω.

In Bayesian games, players condition their actions on their type, i.e., their information on the state. For $i \in N$, a *(mixed) strategy* σ_i of i is a function from T_i to $\Delta(A_i)$, with the probability that action $a_i \in A_i$ is played under strategy σ_i by player i given that his type is $t_i \in T_i$ being denoted by $\sigma_i(a_i \mid t_i)$. Let Σ_i be the set of all strategies of player i. A *strategy profile* is a function $\sigma = (\sigma_i)_{i \in N} \in \times_{i \in N} \Sigma_i$, with for each $i \in N$, σ_i a strategy of player i. As before, payoffs can be extended to mixed strategies.

Let $t_i \in T_i$ be such that $\mu(\tau_i^{-1}(t_i)) > 0$. Then, the *interim expected payoffs* to a player $i \in N$ of type t_i of action $a_i \in A_i$ under prior μ when other players follow the strategy profile $\sigma_{-i} = (\sigma_j)_{j \in N \setminus \{i\}} \in \times_{j \in N \setminus \{i\}} \Sigma_j$ is given by:

$$\varphi_i(a_i, \sigma_{-i}; t_i, \mu) := \sum_{\omega \in \Omega} \mu\Big(\{\omega\} \Big| \tau_i^{-1}(t_i)\Big)\, u_i\Big(a_i, \Big(\sigma_j(\tau_j(\omega))\Big)_{j \in N \setminus \{i\}}, \omega\Big).$$

Definition 2.1.9. *A* Bayesian-Nash equilibrium *is a strategy profile* $\sigma = (\sigma_j)_{j \in N} \in \times_{j \in N} \Sigma_j$ *such that for each* $i \in N$*, for each* $t_i \in T$ *such that* $\mu(\tau_i^{-1}(t_i)) > 0$*, for each* $a_i \in A_i$ *such that* $\sigma_i(a_i \mid t_i) > 0$*,*

$$\varphi_i(a_i, \sigma_{-i}; t_i, \mu) \geq \varphi_i(b_i, \sigma_{-i}; t_i, \mu)$$

for all $b_i \in A_i$*.*

Existence of a Bayesian-Nash equilibrium follows directly from existence of a Nash equilibrium in finite strategic games (e.g. Fudenberg and Tirole, 1991, p. 215).

Definition 2.1.9 requires that each player assigns positive probability to actions that maximize his interim expected payoffs for each type he may end up having, given the strategies of other players. One could also consider a player's ex ante expected payoffs of a strategy profile, i.e., the expected payoff to a player before he learns his type. Formally, the *ex ante expected payoffs* to a player $i \in N$ of a strategy profile $\sigma = (\sigma_i, \sigma_{-i}) \in \times_{j \in N} \Sigma_j$ is given by:

$$\begin{aligned}\Phi_i(\sigma; \mu) &:= \sum_{\omega \in \Omega} \mu(\{\omega\}) u_i(\sigma, \omega) \\ &= \sum_{\substack{t_i \in T_i: \\ \mu(\tau_i^{-1}(t_i)) > 0}} \mu(\tau_i^{-1}(t_i)) \sum_{a_i \in A} \sigma_i(a_i \mid t_i) \varphi_i(a_i, \sigma_{-i}; t_i, \mu).\end{aligned}$$

The following result is standard (e.g. Mas-Colell, Whinston, and Green, 1995, Prop. 8.E.1):

Proposition 2.1.10. *A strategy profile $\sigma = (\sigma_i, \sigma_{-i}) \in \times_{j \in N} \Sigma_j$ is a Bayesian-Nash equilibrium if and only if for each $i \in N$,*

$$\Phi_i(\sigma_i, \sigma_{-i}; \mu) \geq \Phi_i(\sigma_i', \sigma_{-i}; \mu)$$

for any $\sigma_i' \in \Sigma_i$.

2.2 Probability and measure theory

In this section, we cover some basic concepts and results in probability and measure theory, as well as some more specialized concepts and results that we use in this thesis. A good introduction to probability theory is provided by Grimmett and Stirzaker (1992); see e.g. Billingsley (1995) or Dudley (2002) for a more advanced treatment of probability and measure theory.

2.2.1 Basic concepts and definitions

The concept of measure generalizes notions such as length, area and volume. Informally, a measure on a set is an assignment of "sizes" to some subsets of the set that is somehow consistent, in a way to be made precise below. Depending on the application, the size of a subset may for instance be interpreted as area or length, or as the probability that the outcome of an experiment lies in that subset. More specifically, measure theory is used to define integration over general sets. A probability measure is an example of a measure; probability theory builds on measure theory to study random phenomena. In this thesis, we mainly use concepts and results from probability theory, though we sometimes need the more general theory of measure. Here, we therefore introduce both fields, with the emphasis on probability theory.

Given a set, we would like to assign a measure to (some of) the subsets of the set. For reasons beyond the scope of this text,[2] it is not possible to assign a measure to all subsets of a set in a reasonable way when the set is uncountably infinite. Hence, we need to work within subclasses of the class of all subsets of a set. We now define the classes of the appropriate kinds, the algebras and σ-algebras in a set.

2 See Billingsley (1995, pp. 45–46) for a particularly clear formulation of the argument.

Definition 2.2.1. *Given a set S, an* algebra in *S is a collection $\mathscr{F}$ of subsets of S such that:*

(i) $\emptyset \in \mathscr{F}$;
(ii) *if* $E_1, E_2 \in \mathscr{F}$, *then* $E_1 \cup E_2 \in \mathscr{F}$;
(iii) *if* $E \in \mathscr{F}$, *then* $E^c \in \mathscr{F}$.

Definition 2.2.2. *Given a set S, a σ-*algebra in *S is a collection $\mathscr{F}$ of subsets of S such that:*

(i) $\emptyset \in \mathscr{F}$;
(ii) *if* $E_1, E_2, \ldots \in \mathscr{F}$, *then* $\bigcup_{\ell \in \mathbb{N}} E_\ell \in \mathscr{F}$;
(iii) *if* $E \in \mathscr{F}$, *then* $E^c \in \mathscr{F}$.

That is, while algebras are closed under finite unions (and thus, by (*iii*) and DeMorgan laws, under finite intersections) and under complements, σ-algebras are additionally closed under countable unions (and thus under countable intersections). Hence, the class of σ-algebras of a given set is a subset of the class of algebras of that set. As we shall see, measures can be defined on algebras, but often we are interested in measures that are defined on σ-algebras. Simple examples of σ-algebras include the following.

Example 2.2.3. The smallest σ-algebra associated with a set S is the collection $\{\emptyset, S\}$. ◁

Example 2.2.4. Let E be a subset of a set S. Then, the collection $\{\emptyset, E, E^c, S\}$ is a σ-algebra in S. ◁

Example 2.2.5. The *power set* of a set S, i.e., the set of all subsets of S is obviously a σ-algebra. This is the largest σ-algebra in S. ◁

In Example 2.2.4, we explicitly defined a σ-algebra in a set given some subset of this set. More generally, we can generate a σ-algebra in a set starting from some collection of subsets of this set:

Definition 2.2.6. *Given a collection $\mathscr{C}$ of subsets of a set S, the σ-*algebra $\sigma(\mathscr{C})$ generated by *$\mathscr{C}$ is defined to be the smallest σ-algebra in S such that $\mathscr{C} \subseteq \sigma(\mathscr{C})$.*

The collection $\sigma(\mathscr{C})$ is well defined for any collection of subsets $\mathscr{C}$, as the intersection of any nonempty collection of σ-algebras in S is also a σ-algebra in S.

Example 2.2.7. If $\mathscr{C}$ is a σ-algebra in some set S, then obviously $\sigma(\mathscr{C}) = \mathscr{C}$. If $\mathscr{C} \subseteq \mathscr{C}'$, then $\sigma(\mathscr{C}) \subseteq \sigma(\mathscr{C}')$. If $\mathscr{C} \subseteq \mathscr{C}' \subseteq \sigma(\mathscr{C})$, then $\sigma(\mathscr{C}') = \sigma(\mathscr{C})$. ◃

A particularly important σ-algebra is the Borel σ-algebra.

Definition 2.2.8. *Let $\mathscr{I}$ be the collection of all half open intervals $(a, b]$ in $\mathbb{R}$, where $a, b \in \mathbb{R}, a < b$. Then, $\mathscr{B} := \sigma(\mathscr{I})$ is the* Borel σ-algebra. *The elements of $\mathscr{B}$ are called* Borel sets.

The Borel σ-algebra thus forms a σ-algebra in $\mathbb{R}$. More generally, we can define such σ-algebras in $\mathbb{R}^k$:

Definition 2.2.9. *Let $k \in \mathbb{N}$. The σ-algebra $\mathscr{B}^k$ of k-*dimensional Borel sets *is the σ-algebra in $\mathbb{R}^k$ generated by the set of all bounded rectangles in $\mathbb{R}^k$, i.e., by the set of all sets of the form*

$$\left\{x = (x_1, \ldots, x_k) \in \mathbb{R}^k \;\middle|\; a_\ell < x_\ell \leq b_\ell, \ell = 1, \ldots, k\right\},$$

where $a_\ell, b_\ell \in \mathbb{R}, a_\ell < b_\ell$ for $\ell = 1, \ldots, k$.

Definition 2.2.10. *Let Ω be a set. If $\mathscr{F}$ is a σ-algebra in Ω, the pair $(\Omega, \mathscr{F})$ is a* measurable space. *The elements of $\mathscr{F}$ are ($\mathscr{F}$-)*measurable sets *or* events.

Definition 2.2.11. *A function μ on an algebra $\mathscr{F}$ in a set Ω is a* measure on $\mathscr{F}$ *if it satisfies the following conditions:*

(i) $\mu(E) \in [0, \infty]$ *for all* $E \in \mathscr{F}$;
(ii) $\mu(\emptyset) = 0$;
(iii) If $E_1, E_2, \ldots$ are elements of $\mathscr{F}$ such that $E_\ell \cap E_k = \emptyset$ for all $\ell \neq k$ and $\bigcup_{\ell \in \mathbb{N}} E_\ell \in \mathscr{F}$, then

$$\mu\left(\bigcup_{\ell \in \mathbb{N}} E_\ell\right) = \sum_{\ell \in \mathbb{N}} \mu(E_\ell).$$

If $\mu(\Omega) = 1$, then μ is a probability measure.

Condition (*iii*) is referred to as *countable additivity*. It is easy to see that this condition implies finite additivity, which is defined in the obvious way.

An important example of a measure is the Lebesgue measure on $\mathbb{R}$, which is the standard way of assigning a length to an interval. Formally, for $a, b \in \mathbb{R}, a < b$, let

$$\lambda((a, b]) := b - a.$$

Then, λ can be extended to the Borel σ-algebra $\mathscr{B}$ (Billingsley, 1995, p. 168), and it is called the *Lebesgue measure on* $\mathbb{R}$.

Remark 2.2.12. While, as noted above, we cannot assign a measure in a reasonable way to all subsets of a set when the set is uncountable, Definition 2.2.6 allows us to find σ-algebras that contain all subsets of interest. For the important case of the set of real numbers $\mathbb{R}$, the Borel σ-algebra $\mathscr{B}$ contains all relevant intervals, and for $\mathbb{R}^k$, $k \in \mathbb{N}$, we can use the σ-algebra $\mathscr{B}^k$ of k-dimensional Borel sets. ◃

Definition 2.2.13. *Let Ω be a set. If $\mathscr{F}$ is a σ-algebra in Ω and μ is a measure on $\mathscr{F}$, the triple $(\Omega, \mathscr{F}, \mu)$ is a* measure space. *If μ is a probability measure, then we refer to the triple as a probability model or a* probability space.

Throughout this thesis, we mostly focus on probability measures, which we usually denote by $\mathbb{P}$. A probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is a description of an experiment or trial. The set Ω lists all possible outcomes of the experiment, and is referred to as the *sample space*. The probability measure gives for each event $E \in \mathscr{F}$ the probability that E occurs. Hence, all questions and statements associated with an experiment can be formulated in terms of a probability space and vice versa.

Example 2.2.14. In the case of random network models (see Section 2.3), the experiment is a random construction procedure. The random construction procedure determines the probabilities with which all possible networks (the outcomes of the experiment) occur. For instance, given a (finite) set of vertices and some $p \in [0, 1]$, we can construct a random network by drawing an edge between two distinct vertices with probability p, independent of other edges. This defines a probability measure over a set of networks. We discuss this random network model in more detail in Example 2.3.1. ◃

Some more terminology will be useful. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. An event $E \in \mathscr{F}$ that has measure 1 under μ is said to hold (μ-)*almost everywhere*, abbreviated (μ-)a.e. If μ is a probability measure, then we say that E holds (μ-)*almost surely*, abbreviated (μ-)a.s. An event that has probability 0 under μ is called a (μ-)*null* event. Hence, the complement of an almost sure event is a null event.

A probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is *complete* if $A \subseteq B, B \in \mathscr{F}$ and $\mathbb{P}(B) = 0$ together imply that $A \in \mathscr{F}$ and therefore (by countable additivity) that $\mathbb{P}(A) = 0$. Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ that is not complete. It is possible to enlarge the σ-algebra and extend the measure to obtain a probability space that is complete. More specifically, let $\mathscr{F}_0$ be the collection of all subsets of $\mathbb{P}$-null events in $\mathscr{F}$, and let $\mathscr{H} := \sigma(\mathscr{F} \cup \mathscr{F}_0)$ be the smallest σ-algebra that contains the sets in $\mathscr{F}$ and $\mathscr{F}_0$. It can be shown that the domain of $\mathbb{P}$ can be extended in a straightforward way from $\mathscr{F}$ to $\mathscr{H}$ (see Billingsley, 1995, p. 45, for details). Denote the extension of $\mathbb{P}$ to $\mathscr{H}$ by $\mathbb{P}'$. Then, the probability space $(\Omega, \mathscr{H}, \mathbb{P}')$ is complete; it is called the *completion* of $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $E_1, E_2 \in \mathscr{F}$. If $\mathbb{P}(E_1) > 0$, the *conditional probability of E_2 given E_1* is

$$\mathbb{P}(E_2 \mid E_1) := \frac{\mathbb{P}(E_1 \cap E_2)}{\mathbb{P}(E_1)}.$$

Let $\{E_\ell\}_{\ell \in I}$ be a collection of elements of $\mathscr{F}$. Then, the collection of events $\{E_\ell\}_{\ell \in I}$ is *independent* if

$$\mathbb{P}\Big(\bigcap_{\ell \in J} E_\ell\Big) = \prod_{\ell \in J} \mathbb{P}(E_\ell)$$

for any finite subset J of the index set I. Equivalently, we say that the events $\{E_\ell\}_{\ell \in I}$ are independent. Note that if two events E_1, E_2 are independent (and $\mathbb{P}(E_1) > 0$), then $\mathbb{P}(E_2 \mid E_1) = \mathbb{P}(E_2)$. That is, the occurrence or nonoccurrence of the event E_1 does not give any information about the likelihood of the event E_2.

2.2.2 Random variables

Often, we are not so much interested in the experiment itself, but rather in the consequences associated with its random outcome. Random variables and more generally measurable functions provide a means to describe these. Formally, let $(\Omega_1, \mathscr{F}_1)$ and $(\Omega_2, \mathscr{F}_2)$ be two measurable spaces. A function $X : \Omega_1 \to \Omega_2$ is called an *$\mathscr{F}_1/\mathscr{F}_2$-measurable function* if:

$$\{X^{-1}(E) \mid E \in \mathscr{F}_2\} \in \mathscr{F}_1.$$

In words, the inverse image of every $\mathscr{F}_2$-measurable set is $\mathscr{F}_1$-measurable.

When X is a real function, i.e., $\Omega_2 \subseteq \mathbb{R}$, we take $\mathscr{F}_2$ to be the Borel σ-algebra restricted to Ω_2, and we refer to X as a *random variable*. In that case,

$$\forall x \in \Omega_2 : \quad \{\omega \in \Omega_1 \mid X(\omega) \le x\} \in \mathscr{F}_1.$$

When X maps Ω_1 into a subset of $\mathbb{R}^k$, $k \in \mathbb{N}$, $\mathscr{F}_2$ is taken to be the σ-algebra of k-dimensional Borel sets restricted to the subset, and we refer to X as a *random* *vector*. In that case, X is of the form

$$X = (X_1, \dots, X_k)$$

for some real-valued functions $X_1, \dots, X_k$. It can be shown (Billingsley, 1995, p. 183) that X is a $\mathscr{F}_1/\mathscr{B}^k$-measurable function if and only if

$$\forall (x_1, \dots, x_k) \in \Omega_2 : \quad \left\{\omega \in \Omega_1 \;\middle|\; X_1(\omega) \le x_1, \dots, X_k(\omega) \le x_k\right\} \in \mathscr{F}_1.$$

A function $f : \mathbb{R}^\ell \to \mathbb{R}^k$, where $\ell, k \in \mathbb{N}$, is a *Borel function* if it is $\mathscr{B}^\ell/\mathscr{B}^k$-measurable.

Functions of random variables can be random variables. Let X be a random variable defined on some sample space Ω, and let $g : \mathbb{R} \to \mathbb{R}$ be a function on $\mathbb{R}$. Define $g(X)$ by:

$$\forall \omega \in \Omega : \quad g(X)(\omega) := g(X(\omega)).$$

Hence, $g(X)$ maps Ω into $\mathbb{R}$. It is easy to verify that if

$$\forall B \in \mathscr{B} : \quad g^{-1}(B) \in \mathscr{B}, \tag{2.1}$$

then $g(X)$ is a random variable. Condition (2.1) is satisfied if g is e.g. sufficiently smooth or regular by being continuous or monotonic.

Example 2.2.15. A particularly useful class of random variables is formed by the indicator functions of events. Loosely speaking, an indicator function is a function defined on a set that indicates membership of an element in a subset of that set. Formally, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $E \in \mathscr{F}$ be an event. Then, the *indicator function of E* is the function $\mathbf{1}_E : \Omega \to \{0, 1\}$ defined by:

$$\forall \omega \in \Omega : \quad \mathbf{1}_E(\omega) := \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $\mathbf{1}_E$ is a random variable taking the values 1 and 0 with probabilities $\mathbb{P}(E)$ and $\mathbb{P}(E^c)$, respectively. ◁

We can use random variables to generate σ-algebras:

Definition 2.2.16. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space, and let X be a random variable defined on* Ω*. The* σ-algebra generated by X*, denoted* $\sigma(X)$*, is given by:*

$$\sigma(X) := \left\{X^{-1}(E) \;\middle|\; E \in \mathscr{B}\right\}.$$

It can easily be checked that $\sigma(X)$ satisfies conditions (i) - (iii) in Definition 2.2.2. This definition can naturally be extended to σ-algebras generated by random vectors.

When the measurable space $(\Omega_1, \mathscr{F}_1)$ is endowed with a probability measure $\mathbb{P}$, the probability that a random variable on this measurable space takes a certain value, or, more generally, that the image of a measurable function lies in a certain set, can of course be inferred from the combination of the original probability space $(\Omega_1, \mathscr{F}_1, \mathbb{P})$ and the measurable function. However, it is often convenient to define a probability measure directly for the measurable function. Let $(\Omega_1, \mathscr{F}_1, \mathbb{P})$ be a probability space and let $(\Omega_2, \mathscr{F}_2)$ be a measurable space. Let X be a $\mathscr{F}_1/\mathscr{F}_2$-measurable function. The *law of* X is the probability measure $\mathbb{P}_X$ on $\mathscr{F}_2$ defined by:

$$\forall E \in \mathscr{F}_2 : \quad \mathbb{P}_X(E) := \mathbb{P}\Big(X^{-1}(E)\Big).$$

When X is a random variable, its *(cumulative) distribution function* is the function $F_X : \mathbb{R} \to [0,1]$ defined by:

$$\forall x \in \mathbb{R} : \quad F_X(x) := \mathbb{P}\Big(\{\omega \in \Omega_1 \mid X(\omega) \leq x\}\Big).$$

When $X = (X_1, \ldots, X_k)$ is a random vector, the distribution function of X is defined by:

$$\forall x = (x_1, \ldots, x_k) \in \mathbb{R}^k : \quad F_X(x) := \mathbb{P}\Big(\{\omega \in \Omega_1 \mid X_1(\omega) \leq x_1, \ldots, X_k(\omega) \leq x_k\}\Big).$$

The function F_X is also referred to as the *joint distribution function* of the random variables $X_1, \ldots, X_k$. When it is clear from the context with which random variable or vector X a distribution function is associated, we sometimes omit the subscript X.

A distribution function of a random variable X satisfies the following properties (Grimmett and Stirzaker, 1992, Lemma 2.1.6):

(i) $\lim_{x\to-\infty} F_X(x) = 0, \quad \lim_{x\to+\infty} F_X(x) = 1;$
(ii) if $x < y$, then $F_X(x) \leq F_X(y)$;
(iii) F_X is right-continuous, i.e., $\lim_{h\downarrow 0} F_X(x+h) = F_X(x)$.

We say that a random variable X is *discrete* if the support of the distribution function F_X is a finite or countable subset of $\mathbb{R}$. If X is discrete, the *(probability) mass function* or *distribution* of X is the function f from its support S to $(0,1]$ defined by:

$$\forall x \in S : \qquad f(x) := \mathbb{P}\left(\{\omega \in \Omega \mid X(\omega) = x\}\right),$$

so that $\sum_{x\in S} f(x) = 1$.

A collection $\{X_\ell\}_{\ell\in I}$ of random variables defined on Ω_1 is *independent*, or, equivalently, the random variables $\{X_\ell\}_{\ell\in I}$ are independent, if, given any collection $\{B_\ell\}_{\ell\in I}$ of Borel sets, the collection of events $\{\omega \in \Omega_1 \mid X_\ell(\omega) \in B_\ell\}$ is independent. This is equivalent to requiring that

$$\forall\{\ell_1,\dots,\ell_k\} \subseteq I, \forall x = (x_1,\dots,x_k) \in \mathbb{R}^k : \quad F_X(x) = F_{X_{\ell_1}}(x_1)\cdots F_{X_{\ell_k}}(x_k),$$

where F_X is the distribution function of the random vector $X = (X_{\ell_1},\dots,X_{\ell_k})$.

2.2.3 Integration and expectation

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and let f be a real function on Ω that is $\mathscr{F}/\mathscr{B}$-measurable, where we recall that $\mathscr{B}$ is the Borel σ-algebra. We want to define the integral of f with respect to μ.

To define this integral, we need to deal with sums and products involving infinity. Regarding products involving infinity, we use the following conventions:

$$0\cdot\infty = \infty\cdot 0 = 0,$$
$$x\cdot\infty = \infty\cdot x = \infty \text{ if } x \in (0,\infty),$$
$$\infty\cdot\infty = \infty.$$

Furthermore, for a finite or infinite sequence $x, x_1, x_2, \dots$ in $[0,\infty]$,

$$x = \sum_\ell x_\ell \tag{2.2}$$

means that either of the following is the case:

(*i*) $x = \infty$ and $x_\ell = \infty$ for some ℓ;
(*ii*) $x = \infty$ and $x_\ell < \infty$ for all ℓ, and $\sum_\ell x_\ell$ is a divergent infinite series;
(*iii*) $x < \infty$ and $x_\ell < \infty$ for all ℓ, and (2.2) holds in the usual sense for $\sum_\ell x_\ell$ a finite sum or convergent infinite series.

First suppose that f is nonnegative. For each $k \in \mathbb{N}$, for each partition $\mathscr{E} = \{E_1,\dots,E_k\}$ such that $E_\ell \in \mathscr{F}$ for all $\ell = 1,\dots,k$, define the sum

$$S(\mathscr{E}) := \sum_{\ell=1}^{k}\left[\inf_{\omega\in E_\ell} f(\omega)\right]\mu(E_\ell). \tag{2.3}$$

Note that if $E_\ell = \emptyset$ for some $\ell = 1,\dots,k$, then the infimum in (2.3) is equal to ∞ by convention, but then $\mu(E_\ell) = 0$, so that the contribution of the term to the sum (2.3) is zero. Let $\mathcal{D}$ be the set of all finite partitions or decompositions of Ω into $\mathscr{F}$-measurable sets. Then, the integral of the nonnegative real function f on Ω with respect to μ is defined by:

$$\int_\Omega f\, d\mu := \sup_{\mathscr{E}\in\mathcal{D}} S(\mathscr{E}).$$

Now let f be a general real function on Ω that is $\mathscr{F}/\mathscr{B}$-measurable, i.e., f is not necessarily nonnegative. For all $\omega \in \Omega$, define:

$$f^+(\omega) := \max\{f(\omega), 0\}, \qquad f^-(\omega) := -\min\{f(\omega), 0\}.$$

It is easily verified that the functions f^+ and f^- are nonnegative and $\mathscr{F}/\mathscr{B}$-measurable, and that

$$f = f^+ - f^-.$$

Definition 2.2.17. *Suppose that it is not the case that*

$$\int_\Omega f^+\, d\mu = \int_\Omega f^-\, d\mu = \infty.$$

Then, the integral of f on Ω (with respect to μ) *is given by:*

$$\int_\Omega f\, d\mu := \int_\Omega f^+\, d\mu - \int_\Omega f^-\, d\mu.$$

See Billingsley (1995, Ch. 3) for a discussion of general properties of the integral. We can define the integral for a subset of Ω in the following way. For any $E \in \mathscr{F}$, define the function g_E by:

$$\forall\omega \in \Omega: \quad g_E(\omega) := f(\omega)\cdot \mathbf{1}_E(\omega).$$

Then, the integral of f on the set E is given by:

$$\int_E f\, d\mu := \int_\Omega g_E\, d\mu = \int_\Omega f\, \mathbf{1}_E\, d\mu.$$

If both $\int_E f^+\, d\mu$ and $\int_E f^-\, d\mu$ are finite, then f is (μ)-*integrable on* E. If $E = \Omega$, we simply say that f is (μ)-*integrable*. We sometimes omit the subscript Ω or E to the integral if the domain of integration is clear from the context.

We are mostly concerned with two types of integrals, the Lebesgue integral on the real line $\mathbb{R}$, and the expectation of a random variable. To start with the former, recall that λ is the Lebesgue measure on the Borel σ-algebra $\mathscr{B}$ in $\mathbb{R}$. Let $E \in \mathscr{B}$. Let f be a real function that is $\mathscr{B}/\mathscr{B}$-measurable. Then, f is *Lebesgue integrable* on E if f is λ-integrable on E, and its *Lebesgue integral* $\int_E f d\lambda$ is denoted by

$$\int_E f(x)\, dx.$$

It can be shown that

$$\forall u, v \in \mathbb{R}, u < v: \quad \int_{(u,v)} f(x)\, dx = \int_{[u,v)} f(x)\, dx = \int_{(u,v]} f(x)\, dx = \int_{[u,v]} f(x)\, dx,$$

so that if E is an interval, i.e., E is equal to $(u, v), [u, v), (u, v]$ or $[u, v]$ for some $u, v \in \mathbb{R}, u < v$, we can denote the integral $\int_E f(x)\, dx$ by $\int_u^v f(x)\, dx$ without risk of ambiguity. Similarly, when E is equal to $[a, \infty)$ or (a, ∞) for $a \in \mathbb{R}$, we write $\int_a^\infty f(x)\, dx$, etcetera.

We now turn to the expectation of a random variable. The expectation of a random variable is simply the integral of the random variable—a measurable function—with respect to the probability measure of the probability space.

Definition 2.2.18. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let X be a random variable on Ω. Suppose that it is not the case that*

$$\int_\Omega X^+\, d\mathbb{P} = \int_\Omega X^-\, d\mathbb{P} = \infty.$$

Then, the expectation *of X on $(\Omega, \mathscr{F}, \mathbb{P})$, denoted $\mathbb{E}[X]$, is the integral of X with respect to the measure $\mathbb{P}$, i.e.,*

$$\mathbb{E}[X] := \int_\Omega X d\mathbb{P}. \tag{2.4}$$

A random variable X on Ω such that $\mathbb{E}[\,|X|\,] < \infty$ is called *integrable under* $(\Omega, \mathscr{F}, \mathbb{P})$. We refer to $\mathbb{E}$ as the *expectation operator*. The integral $\int_E X d\mathbb{P}$ over a set $E \in \mathscr{F}$ is defined by $\mathbb{E}[X\mathbf{1}_E]$, analogous to the case of general measurable functions.

It can readily be verified that the expectation operator is linear. Let $X_1, \ldots, X_k$ be a sequence of random variables such that $\mathbb{E}[\,|X_\ell|\,] < \infty$ for all $\ell = 1, \ldots, k$. Then, for any $a_1, \ldots, a_k \in \mathbb{R}$,

$$\mathbb{E}\left[\sum_{\ell=1}^k a_\ell X_\ell\right] = \sum_{\ell=1}^k a_\ell \mathbb{E}[X_\ell].$$

The expectation operator has some important continuity properties which we discuss in Section 2.2.6.

Example 2.2.15 (continued). Consider the indicator function $\mathbf{1}_E$ of the event E again. The expectation of $\mathbf{1}_E$ is $\mathbb{P}(E)$, and its variance is $\mathbb{P}(E)(1 - \mathbb{P}(E))$. ◃

We now turn to an important special case of (2.4). Let X be a random variable with distribution function F. The function F gives rise to a probability measure μ'_F on the Borel sets of $\mathbb{R}$ in the following way:

(i) for all $a, b \in \mathbb{R}, a < b$, define $\mu'_F((a, b]) := F(b) - F(a)$;
(ii) extend the domain of μ'_F to include the Borel σ-algebra $\mathscr{B}$.

Then, the triple $(\mathbb{R}, \mathscr{B}, \mu'_F)$ is a probability space. The completion of this probability space is denoted by $(\mathbb{R}, \mathscr{B}_F, \mu_F)$, where $\mathscr{B}_F$ is the smallest σ-algebra containing $\mathscr{B}$ and all subsets of μ'_F-null events, and μ_F is the extension of μ'_F to the domain $\mathscr{B}_F$. We refer to μ_F as the *Lebesgue-Stieltjes measure associated with* F.

Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is $\mathscr{B}_F$-measurable, i.e., condition (2.1) is satisfied. Then, $g(X)$ is a random variable, and the integral

$$\int g \, d\mu_F, \quad \text{denoted by} \quad \int g(x) \, dF(x),$$

is the *Lebesgue-Stieltjes integral* of g with respect to μ_F, so that

$$\mathbb{E}[g(X)] = \int g(x) \, dF(x).$$

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let X be a random variable on Ω with distribution function F_X. Then, the distribution function F_X has *(probability) density function* f (with respect to Lebesgue measure) if f is a nonnegative Borel function on $\mathbb{R}$, and

$$\forall E \in \mathscr{B}: \quad \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in E\}) = \int_E f(x) \, dx. \tag{2.5}$$

We say that f is a (probability) density function for F_X. In particular, by setting $E = \mathbb{R}$,

$$\int_{\mathbb{R}} f(x) \, dx = 1.$$

It can be shown (using Thm. 3.3 of Billingsley, 1995) that (2.5) holds for every Borel set E if it holds for every interval, i.e., if

$$\forall a, b \in \mathbb{R}, a < b: \quad F_X(b) - F_X(a) = \int_a^b f(x)\, dx.$$

Moreover, note that f is determined only to within a set of Lebesgue measure 0: if $f = g$ except on a set of Lebesgue measure 0, then g is also a density function for F_X.

The distribution function F_X has a density function if and only F_X is absolutely continuous. In that case, we say that X is *continuous*. If F_X is absolutely continuous, it is almost everywhere differentiable, and its derivative can be used as its density function. Note that the distribution function of a discrete random variable does not admit a density function.

We close this section with various definitions related to the expectations of random variables that we will use throughout this thesis. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let X be a random variable defined on Ω. For $k \in \mathbb{N}$, the *k-th moment of* X is the value $\mathbb{E}[X^k]$ if the expectation exists. The first moment of X is also called the *mean of* X. It follows immediately from the definitions that if $X > 0$ (almost surely), then $\mathbb{E}[X^k] < \infty$ for $k \in \mathbb{N}$ implies $\mathbb{E}[X^{k-1}] < \infty$. Finally, a related operator that will prove useful is the variance operator. The *variance* of a random variable X is defined as:

$$\mathrm{Var}[X] := \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right].$$

2.2.4 Conditional expectation

We can define the conditional expectation of a random variable given an event:

Definition 2.2.19. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let X be a random variable defined on Ω. Let $E \in \mathscr{F}$ such that $\mathbb{P}(E) > 0$. Then, the* conditional expectation of X given E *is given by:*

$$\mathbb{E}[X \mid E] := \frac{\int_E X d\mathbb{P}}{\mathbb{P}(E)}.$$

More importantly, we can also condition on σ-algebras:

Definition 2.2.20. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space, and let* X *be a random variable defined on* Ω. *Let* $\mathscr{H} \subseteq \mathscr{F}$ *be a* σ*-algebra. Then, the* conditional expectation of X given $\mathscr{H}$ *is a function* Y *from* Ω *to* $\mathbb{R}$ *that satisfies:*

(i) Y *is* $\mathscr{H}$*-measurable;*
(ii) it holds that

$$\forall H \in \mathscr{H}: \quad \int_H Y d\mathbb{P} = \int_H X d\mathbb{P}.$$

If such a function exists, we denote it by $\mathbb{E}[X \mid \mathscr{H}]$.

Some remarks are in order. Firstly, one may be concerned that a conditional expectation may not exists for a given random variable and σ-algebra, or when it does, that it is not unique. However, we have the following result (Dudley, 2002, Thm. 10.1.1):

Theorem 2.2.21. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space, and let* X *be a random variable defined on* Ω. *Let* $\mathscr{H} \subseteq \mathscr{F}$ *be a* σ*-algebra. If* X *is integrable under* $(\Omega, \mathscr{F}, \mathbb{P})$, *then a conditional expectation of* X *given* $\mathscr{H}$ *exists, and any two conditional expectations* Y_1 *and* Y_2 *are equal* $\mathbb{P}$*-almost surely.*

Secondly, note that the conditional expectation of a random variable given a σ-algebra is itself a random variable. In particular, if X is measurable with respect to $\mathscr{H}$ (for instance when $\mathscr{H} = \mathscr{F}$), then X itself satisfies the definition of $\mathbb{E}[X \mid \mathscr{H}]$, and $\mathbb{E}[X \mid \mathscr{H}] = X$, $\mathbb{P}$-almost surely. The case that X is not measurable with respect to $\mathscr{H}$ is of course more interesting.

We are particularly interested in conditional expectations given a certain class of σ-algebras:

Definition 2.2.22. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space, and let* X, Y *be two random variables defined on* Ω *with* X *integrable under* $(\Omega, \mathscr{F}, \mathbb{P})$. *The* conditional expectation of X given Y, *denoted by* $\mathbb{E}[X \mid Y]$, *is the conditional expectation of* X *given the* σ*-algebra* $\sigma(Y)$ *generated by* Y.

When we take the conditional expectation of an indicator function of an event given a σ-algebra, we obtain the probability of the event conditional on the σ-algebra.

Definition 2.2.23. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space, and let* $F \in \mathscr{F}$, *with* $\mathbf{1}_F$ *its indicator function. Let* $\mathscr{H} \subseteq \mathscr{F}$ *be a* σ*-algebra. Then, the* conditional probability of F

given $\mathscr{H}$ *is defined as*

$$\mathbb{P}(F \mid \mathscr{H}) := \mathbb{E}[\mathbf{1}_F \mid \mathscr{H}]. \qquad \square$$

Again, we can condition on the class of σ-algebras generated by random variables:

Definition 2.2.24. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space, and let* $F \in \mathscr{F}$, *with* $\mathbf{1}_F$ *its indicator function. Let* Y *be a random variable defined on* Ω. *Then, the* conditional probability of F given Y, *denoted* $\mathbb{P}(F \mid Y)$, *is defined as* $\mathbb{E}[\mathbf{1}_F \mid Y]$

Note that $\mathbb{P}(F \mid \mathscr{H})$ and $\mathbb{P}(F \mid Y)$ are themselves random variables.

Since the conditional expectation of a random variable given another random variable is itself a random variable, we can take its expectation. It follows directly from the definitions that for any two random variables X, Y with X integrable under $(\Omega, \mathscr{F}, \mathbb{P})$ that

$$\mathbb{E}\Big[\mathbb{E}[X \mid Y]\Big] = \mathbb{E}[X],$$

a result that will prove useful in the following.

2.2.5 Convergence of random variables

We now turn to the topic of convergence of sequences of random variables. We focus here on three principal modes of convergence.

Definition 2.2.25. *Let* $X, X_1, X_2, \dots$ *be random variables on a common probability space* $(\Omega, \mathscr{F}, \mathbb{P})$. *Then, the sequence* $X_1, X_2, \dots$ *converges* almost surely *to* X, *denoted* $X_\ell \xrightarrow{\text{a.s.}} X$, *if*[3]

$$\{\omega \in \Omega \mid X_\ell(\omega) \to X(\omega) \text{ as } \ell \to \infty\}$$

is an event that has probability 1.

Definition 2.2.26. *Let* $X, X_1, X_2, \dots$ *be random variables on a common probability space* $(\Omega, \mathscr{F}, \mathbb{P})$. *Then, the sequence* $X_1, X_2, \dots$ *converges to* X in probability, *denoted* $X_\ell \xrightarrow{\mathbb{P}} X$, *if*

$$\forall \varepsilon > 0: \quad \mathbb{P}\Big(\{\omega \in \Omega \mid |X_\ell(\omega) - X(\omega)| > \varepsilon\}\Big) \to 0$$

as $\ell \to \infty$.

[3] The set on which a sequence of random variables converges is measurable, see e.g. Dudley (2002, p. 127).

Definition 2.2.27. *Let $X, X_1, X_2, \ldots$ be random variables. Then, the sequence $X_1, X_2, \ldots$ converges* in distribution *to X, denoted $X_\ell \xrightarrow{\mathrm{d}} X$, if for all $x \in \mathbb{R}$ such that F_X is continuous at x,*

$$F_{X_\ell}(x) \to F_X(x)$$

as $\ell \to \infty$.

Some remarks are in order. Firstly, it can be shown (Grimmett and Stirzaker, 1992, Thm. 7.2.3) that almost sure convergence implies convergence in probability, which in turn implies convergence in distribution. Convergence in distribution is a condition only on the distribution functions of the random variables. In particular, it contains no reference to the underlying probability space. Hence, convergence in distribution can be defined for random variables that are not even defined on the same probability space. Secondly, it can be shown that convergence of a sequence of random variables in probability to a constant $x \in \mathbb{R}$ is equivalent to convergence of their laws to a point mass (of weight 1) at x (e.g. Grimmett and Stirzaker, 1992, Thm. 7.2.4).

A useful criterion to test whether a sequence of random variables taking values in the set of nonnegative integers $\mathbb{N}_0$ converges in distribution to a certain limit can be derived using probability generating functions, which we define now.[4]

Definition 2.2.28. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $X : \Omega \to \mathbb{N}_0$ be a random variable. For $\ell \in \mathbb{N}_0$, let*

$$q_\ell^{(X)} := \mathbb{P}\left(\{\omega \in \Omega \mid X(\omega) = \ell\}\right).$$

Then, the probability generating function of *X, denoted G_X, is given by:*

$$G_X(t) := \mathbb{E}\left[t^X\right] = \sum_{\ell \in \mathbb{N}_0} q_\ell^{(X)}\, t^\ell \quad \text{for all } t \in \mathbb{R} \text{ such that the sum converges.}$$

Example 2.2.29 (Bernoulli random variables). Suppose X is a random variable on some sample space Ω with

$$\mathbb{P}(\{\omega \in \Omega \mid X(\omega) = 1\}) = p, \qquad \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = 0\}) = 1 - p,$$

where $p \in [0, 1]$. We say that X has the Bernoulli distribution with parameter p. Then,

$$G_X(t) = \mathbb{E}\left[t^X\right] = (1 - p) + pt \tag{2.6}$$

[4] Other criteria for more general classes of random variables also exist; see e.g. Van der Hofstad (2007, Ch. 2).

for all $t \in \mathbb{R}$ such that the sum converges.

A well-known result is that if two random variables X and Y are independent, then (e.g. Grimmett and Stirzaker, 1992, Thm. 5.1.23)

$$G_{X+Y}(t) = G_X(t)G_Y(t) \tag{2.7}$$

for all $t \in \mathbb{R}$ such that $G_X(t)$ and $G_Y(t)$ are well defined.

Example 2.2.30 (Binomially distributed random variables). Let $X_1, \dots, X_k$ be independent Bernoulli random variables on some sample space Ω with parameter $p \in [0,1]$, and define $S := X_1 + \cdots + X_k$. Then, using (2.6) and applying (2.7) repeatedly, we obtain the probability generating function of S:

$$G_S(t) = \prod_{\ell=1}^{k} G_{X_\ell}(t) = \Big((1-p) + pt\Big)^k$$

for all $t \in \mathbb{R}$ such that the sum converges. The random variable S is said to have the binomial distribution with parameters k and p. ◃

A result that we will use extensively in Chapter 5 is the following (e.g. Rotar, 1998, Thm. 11.6.2):

Theorem 2.2.31. *Let $X, X_1, X_2, \dots$ be random variables taking values in $\mathbb{N}_0$, with probability generating functions $G_X, G_{X_1}, G_{X_2}, \dots$. Then,*

$$\forall k \in \mathbb{N}_0 : \quad q_k^{(X_\ell)} \to q_k^{(X)} \text{ as } \ell \to \infty,$$

if and only if

$$G_{X_\ell}(t) \to G_X(t) \text{ as } \ell \to \infty.$$

for all $t \in [0,1]$ such that $G_X(t)$ is well defined. That is, the sequence $X_1, X_2, \dots$ converges to X in distribution if and only if the sequence of the associated generating functions converges to the probability generating function of X.

Often, we are interested in the convergence of sums of random variables.

Definition 2.2.32. *Let $X_1, X_2, \dots$ be a sequence of random variables on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with partial sums $S_k := \sum_{\ell=1}^{k} X_\ell$. The sequence $X_1, X_2, \dots$ obeys the* weak law of large numbers *if there exists $x \in \mathbb{R}$ such that*

$$k^{-1} S_k \xrightarrow{\text{P}} x \quad \text{when } k \to \infty.$$

The sequence obeys the strong law of large numbers *if there exists* $x \in \mathbb{R}$ *such that*

$$k^{-1} S_k \xrightarrow{\text{a.s.}} x \quad \text{when } k \to \infty.$$

Obviously, when a sequence of random variables satisfies the strong law of large numbers, it satisfies the weak law of large numbers. One may try to identify sufficient, and if possible necessary, conditions on sequences of random variables for the weak and the strong law of large numbers to hold. Here, we only use the following result, which is commonly and somewhat confusingly (cf. Definition 2.2.32) referred to as the strong law of large numbers (Dudley, 2002, Thm. 8.3.5).

Theorem 2.2.33 (Strong law of large numbers). *Let* $X_1, X_2, \ldots$ *be a sequence of independent and identically distributed random variables on some probability space* $(\Omega, \mathscr{F}, \mathbb{P})$. *Then, when* $k \to \infty$,

$$\frac{1}{k} \sum_{\ell=1}^{k} X_\ell \xrightarrow{\text{a.s.}} x \quad \text{for some } x \in \mathbb{R}$$

if and only if $\mathbb{E}[\,|X_1|\,] < \infty$. *In that case,* $x = \mathbb{E}[X_1]$.

Finally, a useful result is Markov's inequality:

Lemma 2.2.34 (Markov's inequality). *Let* X *be a random variable with finite first moment defined on some sample space* Ω. *Then,*

$$\forall a > 0: \quad \mathbb{P}\Big(\{\omega \in \Omega \mid |X(\omega)| \geq a\}\Big) \leq \frac{1}{a}\,\mathbb{E}[\,|X|\,].$$

Proof. Define $A := \{\omega \in \Omega \mid |X(\omega)| \geq a\}$. Then, $|X(\omega)| \geq a\,\mathbf{1}_A(\omega)$ for all $\omega \in \Omega$. Taking expectations gives the desired result. □

We sometimes use the somewhat stronger result below.

Lemma 2.2.35. *Let* X *be a random variable with finite first moment defined on some sample space* Ω. *Then,*

$$\forall a > 0: \quad \mathbb{P}\Big(\{\omega \in \Omega \mid |X(\omega)| \geq a\}\Big) \leq \frac{1}{a}\,\mathbb{E}[X\,\mathbf{1}_{\{\omega \in \Omega \mid |X(\omega)| \geq a\}}].$$

Proof. Again, define $A := \{\omega \in \Omega \mid |X(\omega)| \geq a\}$. Then, $X(\omega)\,\mathbf{1}_A(\omega) \geq a\,\mathbf{1}_A(\omega)$ for all $\omega \in \Omega$. Taking expectations gives the desired result. □

2.2.6 Continuity of the expectation operator

The expectation operator satisfies some useful continuity properties (Billingsley, 1995):

Lemma 2.2.36. *Let $X, X_1, X_2, \ldots$ be random variables.*

- **Monotone convergence:** *Suppose that $X, X_1, X_2, \ldots$ are all defined on the same probability space and suppose that $X_\ell \xrightarrow{\text{a.s.}} X$. If for all $\ell \in \mathbb{N}$, $X_\ell \geq 0$ a.s. and $X_\ell \leq X_{\ell+1}$ a.s., then*

$$\mathbb{E}[X_\ell] \to \mathbb{E}[X] \quad \textit{when } \ell \to \infty.$$

- **Dominated convergence:** *Suppose $X_\ell \xrightarrow{\text{d}} X$. If there exists a random variable Y with $\mathbb{E}[\,|Y|\,] < \infty$ such that*

$$\forall \ell \in \mathbb{N}, x > 0 : \quad 1 - F_{|X_\ell|}(x) \leq 1 - F_{|Y|}(x),$$

then $\mathbb{E}[\,|X|\,] < \infty$ and

$$\mathbb{E}[X_\ell] \to \mathbb{E}[X] \quad \textit{when } \ell \to \infty.$$

- **Bounded convergence:** *Suppose $X_\ell \xrightarrow{\text{d}} X$. If there exists $x \in \mathbb{R}$ such that $|X_\ell| \leq x$ a.s., then $\mathbb{E}[\,|X|\,] < \infty$ and*

$$\mathbb{E}[X_\ell] \to \mathbb{E}[X] \quad \textit{when } \ell \to \infty.$$

Note that bounded convergence is in fact a special case of dominated convergence.

Remark 2.2.37. It is customary in probability theory to omit the argument ω. In particular, $[X \in B]$ is often used as shorthand for $\{\omega \in \Omega \mid X(\omega) \in B\}$. The probability of this event is denoted by $\mathbb{P}(X \in B)$ (without the square brackets), and its indicator function is written as $\mathbf{1}_{[X \in B]}$. For the remainder of this thesis, we will follow this convention for ease of notation, unless confusion may arise. $\triangleleft$

2.3 Random networks

Random network models are a special class of probability spaces. The theory of random networks can therefore be seen as a subfield of probability theory. However, given its central place in Part I of this thesis and because it is also intimately related to other fields such as graph theory, we treat the theory of random networks separately here. In this section, we first introduce the main concepts and definitions. Subsequently, we discuss a class of random network models that has recently received considerable interest in economics, the class of random network models with a given degree distribution.

Books on random networks in mathematics generally cover a limited number of random network models in depth. Bollobás (2001) and Janson et al. (2000) are classic references for the Erdős-Rényi random network models that we discuss shortly. Durrett (2006), Jackson (2008) and Van der Hofstad (2007) additionally discuss some other well-known random network models. Bollobás and Riordan (2003) offer a survey of rigorous results on the so-called preferential attachment model of Barabási and Albert (1999). The theory of random networks uses concepts and results from graph theory; a good introduction to this field is provided by West (2001).

2.3.1 Basic concepts and definitions

Before we can define random networks, we need to specify what we mean by the term network. A *network* g is a pair consisting of a finite, nonempty set $V(g)$ of *vertices* and a finite set $E(g)$ of *edges*, with an edge being an unordered pair of two distinct vertices. Let g be a network. If $\{v, w\} \in E(g)$, where $v, w \in V(g), v \neq w$, then v and w are *neighbors* in g; alternatively, we say that v and w are *adjacent* in g. For ease of notation, an edge $\{v, w\} \in E(g)$ is sometimes denoted by vw. An *independent set* in a network is a set of vertices that are pairwise nonadjacent. A network is *bipartite* if its vertex set is the union of two disjoint (possibly empty) independent sets. This is illustrated in Figure 2.4. Finally, two networks g, g' are *isomorphic* if $V(g) = V(g') =: V$ and there is a permutation π of V such that $\{i, j\} \in E(g)$ for $i, j \in V, i \neq j$, if and only if $\{\pi(i), \pi(j)\} \in E(g')$. This defines an equivalence relation; hence, the set of all networks with a given vertex set can be partitioned into a finite number of *isomorphism classes*, i.e., sets of isomorphic networks.

We now turn to random networks. A *random network model* is a probability space (i.e., a triple consisting of a sample space, a σ-algebra, and a probability measure on the σ-algebra) in which the sample space is a nonempty set of networks. This set can be finite or countable. The probability measure on the σ-algebra is determined by a random construction process, the experiment. The outcome of such an experiment is called a *random network*.

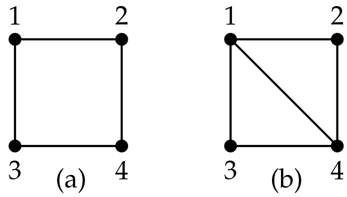


Figure 2.4. (a) A bipartite network, with independent sets $\{1, 4\}$ and $\{2, 3\}$; (b) a network that is not bipartite: while vertex 2 and 3 are not connected, vertex 1 and 4 are now neighbors, so there do not exist two independent sets that partition the vertex set $\{1, 2, 3, 4\}$.

Example 2.3.1 (Erdős-Rényi random networks). The notion of a random network originated in a paper by Erdős (1947). In the random network model proposed by Erdős (1947), a network is chosen uniformly at random from the set of all $2^{\binom{n}{2}}$ networks with n vertices, where $n \in \mathbb{N}$. Hence, the probability space is $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mathbb{P}^{(n)}_{1/2})$, where $\mathcal{G}^{(n)}$ is the set of all networks with n vertices, $\mathscr{F}^{(n)}$ is the set of all subsets of $\mathcal{G}^{(n)}$, and

$$\forall g \in \mathcal{G}^{(n)} : \quad \mathbb{P}^{(n)}_{1/2}(g) := 2^{-\binom{n}{2}}.$$

This probability space can also be interpreted as the result of $\binom{n}{2}$ tosses of a fair coin (hence the subscript $1/2$), one for each pair of vertices, with an edge being drawn between two vertices if the toss gives heads (say). This model can of course be generalized: we can draw an edge between each pair of vertices with probability $p = p(n)$, rather than with probability $1/2$ (Gilbert, 1959). In that case,

$$\forall g \in \mathcal{G}^{(n)} : \quad \mathbb{P}^{(n)}_{p}(g) = (p)^{m_g} (1-p)^{\binom{n}{2} - m_g},$$

where m_g is the number of edges in g. Erdős and Rényi (1959) proposed the closely related random network model consisting of all networks with n vertices and $M = M(n)$ edges, with all networks having equal probability. For many purposes, the latter two models are essentially equivalent for appropriate choices of p and M. As Erdős and Rényi are generally recognized to be the founders of the theory of random networks, these models are commonly referred to as *Erdős-Rényi random network models*. ◃

Let $n \in \mathbb{N}$, and let $V^{(n)} := \{1, \ldots, n\}$. Let $\mathcal{G}^{(n)}$ be the set of all networks with vertex set $V^{(n)}$, and let $\mathscr{F}^{(n)}$ be the set of all subsets of $\mathcal{G}^{(n)}$. Let

$$\mathcal{G} := \bigcup_{n \in \mathbb{N}} \mathcal{G}^{(n)}$$

be the countable set of all networks with a finite vertex set. Define

$$\mathcal{V} := \bigcup_{n \in \mathbb{N}} V^{(n)} = \mathbb{N}.$$

Let $\mathscr{F}$ be the σ-algebra generated by the set of singletons of $\mathcal{G}$.

We define some random variables that will be useful in the following. Note that if a function is $\mathscr{F}/\mathscr{H}$-measurable for some σ-algebra $\mathscr{H}$, the function with its domain restricted to $\mathcal{G}^{(n)}$ is $\mathscr{F}^{(n)}/\mathscr{H}$-measurable for all $n \in \mathbb{N}$. Hence, if we define a measurable function on $\mathcal{G}$, it is defined on $\mathcal{G}^{(n)}$ for all $n \in \mathbb{N}$. Let $\mathcal{Q}$ be the (countable) set of all finite subsets of $\mathcal{V}$.

We are interested in the local environment of vertices. Let $v \in \mathcal{V}$, and define the function $N_v : \mathcal{G} \to \mathcal{Q}$ by:

$$\forall g \in \mathcal{G} : \quad N_v(g) := \Big\{ w \in V(g) \;\Big|\; vw \in E(g) \Big\}.$$

Hence, $N_v(g)$ is the set of neighbors of vertex v in network g. We refer to the measurable function N_v as the *neighborhood of v*, and to $N_v(g)$, $g \in \mathcal{G}$, as the *neighborhood of v in g*. Also, define the function $D_v : \mathcal{G} \to \mathbb{N}_0$ by:

$$\forall g \in \mathcal{G} : \quad D_v(g) := |N_v(g)|.$$

That is, $D_v(g)$ is the number of neighbors of vertex v in network g. We refer to $D_v(g)$ as the *degree of v in g*, and to the random variable D_v as the *degree of v*. Note that the degree of v in g can be 0 for two distinct reasons. It could be that v is a vertex in the network, but does not have any neighbors, or that v is not a vertex of the network.

We also consider the number of neighbors the neighbors of a given vertex have. Loosely speaking, the neighbor degree profile of a vertex in a given network is a list of the degrees of the neighbors of the vertex, in a non-increasing order. For $t \in \mathbb{N}$, let

$$\Omega_K^t := \{(k_1, \ldots, k_t) \in \mathbb{N}^t \mid k_1 \geq k_2 \geq \ldots \geq k_{t-1} \geq k_t\}.$$

For $t = 0$, let $\Omega_K^t := \{0\}$, and define

$$\Omega_K := \bigcup_{t \in \mathbb{N}_0} \Omega_K^t.$$

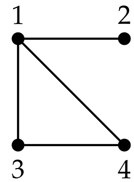


Figure 2.5. The network g of Example 2.3.2.

Let $\mathscr{F}_K$ be the σ-field generated by the set of singletons of Ω_K. For $g \in \mathcal{G}$ and $v \in \mathcal{V}$ such that $D_v(g) = 0$, we set $K_v(g) := 0$. Otherwise, define

$$\begin{aligned} N_1 &:= N_v(g), \\ j(1) &:= \max\{w \in N_1 \mid D_w(g) \geq D_z(g) \text{ for all } z \in N_1\}, \\ K_{v,1}(g) &:= D_{j(1)}(g), \end{aligned}$$

and for $\ell = 2, \ldots, D_v(g)$:

$$\begin{aligned} N_\ell &:= N_{\ell-1} \setminus \{j(\ell - 1)\}, \\ j(\ell) &:= \max\{w \in N_\ell \mid D_w(g) \geq D_z(g) \text{ for all } z \in N_\ell\}, \\ K_{v,\ell}(g) &:= D_{j(\ell)}(g). \end{aligned}$$

Then, $K_v(g) := (K_{v,1}(g), \ldots, K_{v,D_v(g)}(g))$ is the *neighbor degree profile of v in g*, and the function $K_v : \mathcal{G} \to \Omega_K$ is the *neighbor degree profile of v*.

Example 2.3.2. Suppose we draw network g in Figure 2.5 from the set $\mathcal{G}$. Its vertex set is $V(g) = \{1, 2, 3, 4\}$, and its edge set is $E(g) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 4\}\}$. The neighborhood of vertex 1 in g is $N_1(g) = \{2, 3, 4\}$, and its degree in g is $D_1(g) = 3$. The neighbor degree profile of vertex 1 in g is $K_1(g) = (D_4(g), D_3(g), D_2(g)) = (2, 2, 1)$. $\triangleleft$

Throughout this thesis, we make the following assumption:

Assumption 2.A (Finite expected number of vertices). Let $(\mathcal{G}, \mathscr{F}, \mathbb{P})$ be a random network model. The expected number of vertices in $(\mathcal{G}, \mathscr{F}, \mathbb{P})$ is finite, i.e.,

$$\sum_{n \in \mathbb{N}} n \mathbb{P}\Big(\mathcal{G}^{(n)}\Big) < \infty. \qquad \triangleleft$$

Random network models with this property exist. For instance, any random network model with $\mathbb{P}(\mathcal{G}^{(n)}) = 1$ for some $n \in \mathbb{N}$ satisfies this property. The random network model in Example 4.9 of Bollobás et al. (2007) is an instance of a random network model with a random number of vertices that satisfies this property.

Remark 2.3.3. We have defined a random network model as a probability space, where the sample space is (a subset of) $\mathcal{G}$. An alternative approach is to define a random network model as a measurable function from some underlying probability space to (a subset of) $\mathcal{G}$. The two approaches are equivalent (Billingsley, 1999, p. 25). We have chosen to outline the former approach here as it is the more natural one for defining network belief systems in the context of network games (Chapter 3 and 4). We pursue the latter approach in Chapter 5. ◁

Often, one is interested in random network models with a fixed number of vertices in which vertices are ex ante identical in terms of their neighbor degree profile:

Definition 2.3.4. *Let $(C^{(n)}, \mathscr{H}^{(n)}, \mathbb{P})$ be a random network model such that $C^{(n)} \subseteq \mathcal{G}^{(n)}$ for some $n \in \mathbb{N}$ and $\mathscr{H}^{(n)}$ the σ-algebra generated by the set of singletons of $C^{(n)}$. The neighbor degree profiles $K_1, K_2, \ldots, K_n$ are* exchangeable under $(C^{(n)}, \mathscr{H}^{(n)}, \mathbb{P})$ *if for any $k \in \{1, \ldots, n\}$, any $v_1, \ldots, v_k \in V^{(n)}$, the random vector $(K_{v_1}, K_{v_2}, \ldots, K_{v_k})$ has the same distribution as $(K_{\pi(v_1)}, K_{\pi(v_2)}, \ldots, K_{\pi(v_k)})$ for any permutation $\pi : \{v_1, \ldots, v_k\} \to \{v_1, \ldots, v_k\}$.*

Examples of a random network model with a fixed number of vertices and exchangeable neighbor degree profiles are discussed in Section 2.3.2 and in Chapters 3–5. Note that if neighbor degree profiles are exchangeable under a random network model $(C^{(n)}, \mathscr{H}^{(n)}, \mathbb{P})$ where $C^{(n)} \subseteq \mathcal{G}^{(n)}$ for some $n \in \mathbb{N}$, then in particular, for any $v, w \in V^{(n)}$ and for each $t \in \mathbb{N}_0$,

$$\mathbb{P}\Big(\{g \in C^{(n)} \mid D_v(g) = t\}\Big) = \mathbb{P}\Big(\{g \in C^{(n)} \mid D_w(g) = t\}\Big),$$

i.e., the probability that a vertex has a certain degree is the same for each vertex. This means that if we want to consider the probability that an arbitrary vertex has a given degree $t \in \mathbb{N}_0$, we can simply consider the probability that a fixed vertex, say vertex 1, has degree t. This we use in the following definition.

Definition 2.3.5. *Let $(C^{(n)}, \mathscr{H}^{(n)}, \mathbb{P})$ be a random network model with $C^{(n)} \subseteq \mathcal{G}^{(n)}$ for some $n \in \mathbb{N}$ and $\mathscr{H}^{(n)}$ the σ-algebra generated by the set of singletons of $C^{(n)}$. Suppose neighbor degree profiles are exchangeable under $(C^{(n)}, \mathscr{H}^{(n)}, \mathbb{P})$. Then, the* degree distribution of $(C^{(n)}, \mathscr{H}^{(n)}, \mathbb{P})$ *is given by $\left(p_t^{(n)}\right)_{t \in \mathbb{N}_0}$, where*

$$\forall t \in \mathbb{N}_0 : \quad p_t^{(n)} := \mathbb{P}\Big(\{g \in C^{(n)} \mid D_1(g) = t\}\Big).$$

That is, the degree distribution of a random network model gives for each $t \in \mathbb{N}_0$ the probability that a vertex selected uniformly at random from the network has degree t.

Remark 2.3.6. It is also possible to define the degree distribution for more general classes of random network models. We do not pursue that direction here, and refer the reader to Chapter 4 for an example of how to define the degree distribution in a random network model with a random number of vertices. ◃

Example 2.3.1 (continued). For $n \in \mathbb{N}$, consider the Erdős-Rényi random network model $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mathbb{P}_p^{(n)})$, where $p = p(n)$. It is not hard to see that the probability that a given vertex has exactly k neighbors, where $k \in \{0, 1, \ldots, n-1\}$, is

$$\binom{n-1}{k} p^k (1-p)^k,$$

i.e., the degree of a vertex is a binomial random variable with parameters $n-1$ and p. If we set $p(n) = c/n$ for $n \in \mathbb{N}$, where $c > 0$, the degree distribution of $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mathbb{P}_p^{(n)})$ converges to the Poisson distribution with parameter c when $n \to \infty$ (Van der Hofstad, 2007, Thm. 5.9). ◃

The degree distribution is an important property of random network models. Many networks in economic and social contexts are characterized by power law degree sequences (Newman, 2003b). The degree sequence of a network gives for each t the fraction of vertices with degree t, and is thus the deterministic analogue of the degree distribution.[5] In a network with a power law degree sequence, the fraction of vertices with degree t falls off approximately as $t^{-\alpha}$, where $\alpha > 0$, at least for some range of t. In the case of degree distributions, the interest is merely in the (right) tail behavior, i.e., the probability that a vertex has a very high degree. We say that a cumulative distribution function F is *heavy-tailed* or has a *heavy (right) tail* if for all $a > 0$,

$$\lim_{t \to \infty} e^{at}(1 - F(t)) \to \infty.$$

An example of a heavy-tailed distribution function is a distribution function with a *power law tail*, i.e., a distribution function F such that $(1 - F(t)) \sim t^{-\alpha}$, where $\alpha > 0$ (recall that $\sim$ denotes that two functions are asymptotically equal, see page xv for a precise definition). A prominent example of a distribution with a distribution function with a power law tail is the *Pareto distribution*, whose distribution function

[5] There is no consensus on the definition of a degree sequence. We use the definition from the theory of random networks; in graph theory, the degree sequence of a network is usually taken to be a non-increasing sequence of the degrees of the vertices in the networks, and the term "degree distribution" is used to denote what we call the degree sequence. We use the current term to distinguish between the deterministic concept (degree sequence) and the stochastic concept (degree distribution).

is given by

$$\forall t \geq \beta : \quad F(t) = 1 - \left(\frac{\beta}{t}\right)^{-\alpha},$$

where $\alpha, \beta > 0$. A distribution function has *thin tails* if there exists $a > 0$ such that

$$\lim_{t\to\infty} e^{at}(1 - F(t)) \to 0.$$

We have seen that the degree distribution of a sequence $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mathbb{P}_p^{(n)})_{n\in\mathbb{N}}$ of Erdős-Rényi random network models with $p = c/n$ for $n \in \mathbb{N}$, where $c > 0$, converges to the Poisson distribution with parameter c when $n \to \infty$. As the Poisson distribution is a typical example of a distribution with a distribution function with thin tails, the Erdős-Rényi random network models cannot account for the power law degree sequence observed in many real networks. By contrast, the class of random network models discussed in the next section can generate arbitrary degree distributions, including so-called power law degree distributions. For that reason, it recently has received considerable attention in different fields, including economics (e.g. Jackson, 2008; Vega-Redondo, 2007). In Chapter 5, we propose another random network model that can also generate power law degree distributions.

2.3.2 Random networks with a given degree distribution

We define a class of random network models that can generate arbitrary degree distributions. We use the erased configuration model of Britton et al. (2006). Let ξ be a distribution with support in the set of nonnegative integers. Let $n \in \mathbb{N}$, and consider a random network model $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mathbb{P}_\xi^{(n)})$, where $\mathcal{G}^{(n)}$ is the set of all networks on vertex set $V^{(n)}$, $\mathscr{F}^{(n)}$ is the set of all subsets of $\mathcal{G}$, and the probability measure $\mathbb{P}_\xi^{(n)}$ is defined indirectly as follows.

For each vertex $v \in V^{(n)}$, draw a number H_v independently from the distribution ξ, and attach H_v "half-edges" to v. We want to pair these half-edges in such a way that each half-edge is paired with exactly one other half-edge. Clearly, if the sum $\sum_{v\in V^{(n)}} H_v$ is odd, this is not possible. In that case, we select a vertex w uniformly at random from $V^{(n)}$ and replace H_w by $H_w + 1$. We then select pairs of half-edges uniformly at random (without replacement) and connect the elements of the pair, until no half-edges are left. If we have connected a half-edge from v to a half-edge from w, where $v, w \in V^{(n)}$, we say that there is a *potential edge with* *endpoints v and w*. Note that there can be more than one potential edge with a given pair of vertices as its endpoints, or that there can be potential edges for which the two endpoints are equal. We construct a random network by drawing an edge between two vertices $v, w \in V^{(n)}, v \neq w$, if and only if there is at least one potential edge with v and w as its endpoints.

Remark 2.3.7. The erased configuration model of Britton et al. (2006) we discuss here is a variant of the configuration model (Bender and Canfield, 1978; Bollobás, 1980), which many authors in economics refer to in order to construct random networks. The configuration model is essentially identical to the model presented here, except that *each* potential edge is replaced by an edge.[6] Hence, under the configuration model, there can be multiple edges between a pair of vertices, or there can be self-loops, i.e., edges from a given vertex to itself. For game-theoretical applications, this is undesirable. We therefore use the erased configuration model of Britton et al. which does not allow for multiple edges or self-loops. ◁

This defines the random network model $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mathbb{P}_{\xi}^{(n)})$. Notice that neighbor degree profiles are exchangeable under this random network model. The degree distribution of this random network model is *not* identical to ξ. The reason is that, even though the number of half-edges of each vertex is distributed according to ξ, the degree distribution will differ somewhat from that distribution, because not every potential edge is replaced by an edge.[7] However, when the number of half-edges (distributed as ξ) of each vertex has finite first moment, the degree distribution of $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mathbb{P}_{\xi}^{(n)})$ comes arbitrarily close to ξ when n grows large:

Theorem 2.3.8 (Britton et al. (2006), Thm. 2.1). *For each $n \in \mathbb{N}$, let the number of half-edges of a vertex be distributed as ξ, with support in $\mathbb{N}$, and suppose it has finite first moment. Let $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mathbb{P}_{\xi}^{(n)})$ be defined as above, and let $(p_t^{(n)})_{t \in \mathbb{N}_0}$ be its degree distribution. Then,*

$$\forall t \in \mathbb{N}_0 : \quad p_t^{(n)} \to \xi(t),$$

as $n \to \infty$. That is, the degree distribution of $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mathbb{P}_{\xi}^{(n)})$ converges to ξ when $n \to \infty$.

Proof. See Britton et al. (2006). □

6 The configuration model is often used with a fixed sequence $(h_v)_{v \in V^{(n)}}$ that gives for each vertex $v \in V^{(n)}$ the number of half-edges h_v of v, rather than with an i.i.d. sequence (Bollobás, 2001). The configuration model with a fixed sequence and the configuration model with an i.i.d. sequence are intimately related, see Van der Hofstad (2007).

7 Also, when the sum of half-edges is odd, we increase the number of half-edges by 1 of a vertex selected uniformly at random from the vertex set. In the limit of large n, this effect is negligible (Britton et al., 2006, also see Appendix 2.A).

We refer to $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mathbb{P}_{\xi}^{(n)})$ as the *random network model with n vertices and (asymptotic) degree distribution* ξ. The intuition behind Theorem 2.3.8 is that, even though with positive probability, there are multiple potential edges between *some* vertex and another vertex, or potential edges from *some* vertex to itself,[8] the probability that this is the case for a *given* vertex becomes vanishingly small as the number of vertices grows large (as long as the first moment of the number of half-edges of each vertex is finite).

A similar intuition lies behind the following result, which states that the degrees of neighbors are asymptotically independent when the first moment of the number of half-edges is finite. Before we state this result, first note that if the probability that a vertex selected uniformly at random from the network has degree t is $q_t \in [0,1]$ for $t \in \mathbb{N}_0$, then the probability that a *neighbor* of a vertex selected uniformly at random from the network has degree t is proportional to tq_t, as a vertex of degree t has t times more neighbors than a vertex of degree 1. For each $t \in \mathbb{N}_0$, $\theta = (\theta_1, \ldots, \theta_t) \in \Omega_K^t$ and $k \in \mathbb{N}_0$, define $c_k(\theta)$ to be the number of elements in θ that are equal to k. Define

$$M(\theta) := \frac{t!}{\prod_{k \in \mathbb{N}_0} c_k(\theta)!},$$

where we recall that $0! = 1$. That is, $M(\theta)$ is the multinomial coefficient corresponding to θ. Also, recall that $x^0 = 1$ for $x > 0$.

Theorem 2.3.9. *For each $n \in \mathbb{N}$, let the number of half-edges of a vertex be distributed as ξ, with support in $\mathbb{N}$, and suppose it has finite first moment. Let $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mathbb{P}_{\xi}^{(n)})$ be defined as above. Then, for each $t \in \mathbb{N}$, for each $\theta = (\theta_1, \ldots, \theta_t) \in \Omega_K^t$,*

$$\mathbb{P}_{\xi}^{(n)}(K_1 = \theta) \to \xi(t)\, M(\theta) \prod_{\substack{s \in \mathbb{N}: \\ \xi(s) > 0}} \left(\frac{s\xi(s)}{\sum_{r \in \mathbb{N}} r\xi(r)} \right)^{c_s(\theta)},$$

as $n \to \infty$. That is, the degrees of a vertex and his neighbors are asymptotically independent under $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mathbb{P}_{\xi}^{(n)})$.

Since there is no explicit proof of this result in the literature, as far as we are aware, we give the full proof in Appendix 2.A.[9]

[8] In fact, it can be shown that this probability is bounded away from zero for all $n \in \mathbb{N}$ (Bender and Canfield, 1978).

[9] There are some results on the asymptotic independence of degrees for certain random network models, but these are generally of a different type: it is shown that the degrees of some fixed finite set of vertices are independent. However, for game-theoretic applications, one needs that the degrees of a fixed vertex and its *neighbors* are independent.

In Chapter 3 and 4, we show that the dependencies or independencies among vertex degrees can affect game-theoretic predictions when we study strategic interactions on a network. In Chapter 5, we define a random network model with a given asymptotic degree distribution in which vertex degrees are *not* asymptotically independent.

2.A Proof of Theorem 2.3.9

First we need some more definitions. We first generalize the notion of a network. A *pseudograph* g_P is a triple consisting of a finite, nonempty vertex set $V(g_P)$, a finite set of *pseudo-edges* $E_P(g_P)$,[10] and a relation that associates with each pseudo-edge two vertices (not necessarily distinct) called its *endpoints*. A pseudograph may contain *multiple edges*, i.e., distinct pseudo-edges with the same pair of vertices as its endpoints, or *self-loops*, pseudo-edges whose endpoints are equal. A pseudograph is *simple* if it does not contain multiple edges or self-loops; see Figure 2.6 for an illustration. Hence, a network is a simple pseudograph.[11] Two distinct vertices $v, w \in V(g_P), v \neq w$, are *adjacent* in a pseudograph g_P if there is a pseudo-edge $e \in E_P(g_P)$ with v and w as its endpoints; alternatively, we say that v and w are neighbors.

In Section 2.3.2, we considered a random construction process where for a given probability measure ξ and a given $n \in \mathbb{N}$ potential edges were drawn between pairs of vertices in $V^{(n)} = \{1, \ldots, n\}$. If we identify each potential edge with a pseudo-edge, we obtain a pseudograph. The random construction process, together with the identification of each potential edge with a pseudo-edge, thus defines a probability space $(\mathcal{G}_P^{(n)}, \mathscr{F}_P^{(n)}, \mathbb{P}_{\xi,P}^{(n)})$, where:

- $\mathcal{G}_P^{(n)}$ is the (countable) set of all pseudographs with vertex set $V^{(n)}$;
- $\mathscr{F}_P^{(n)}$ is the σ-algebra generated by the set of singletons of $\mathcal{G}_P^{(n)}$;
- $\mathbb{P}_{\xi,P}^{(n)}$ is the probability measure on $\mathcal{G}_P^{(n)}$ induced by the random construction process.

Define

$$\mathcal{G}_P := \bigcup_{n \in \mathbb{N}} \mathcal{G}_P^{(n)}$$

10 In graph theory, pseudo-edges are simply called edges (e.g. West, 2001). We use the current terminology to avoid confusion with the notion of an edge that we introduced in Section 2.3.2.

11 There is no consensus on definitions here. Some authors define a network (or graph) to be what we call a pseudograph, and refer to what we call a network as a simple network. Others use the term multigraph rather than pseudograph. See Goldberg et al. (2003) for a discussion.

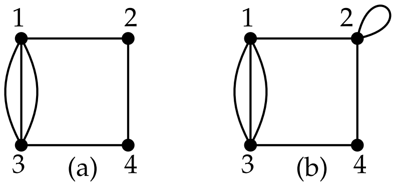


Figure 2.6. (a) A pseudograph with multiple edges; (b) a pseudograph with multiple edges and a self-loop.

to be the set of all finite pseudographs, and let $\mathscr{F}_P$ be the σ-algebra generated by the set of singletons of $\mathcal{G}_P$. Recall the definition of $\mathcal{V}$ and $V^{(n)}$, where $n \in \mathbb{N}$.

We define some random variables that will be useful in the following. Note that if a function is $\mathscr{F}_P/\mathscr{H}$-measurable for some σ-algebra $\mathscr{H}$, the function with its domain restricted to $\mathcal{G}_P^{(n)}$ is $\mathscr{F}_P^{(n)}/\mathscr{H}$-measurable for all $n \in \mathbb{N}$. Let $v \in \mathcal{V}$. The *pseudo-degree* of v is the random variable $D_v^P : \mathcal{G}_P \to \mathbb{N}_0$ defined by:

$$\forall g_P \in \mathcal{G}_P : \quad D_v^P(g_P) := \Big|\{e \in E_P(g_P) \mid v \text{ is an endpoint of } e\}\Big|,$$

i.e., the pseudo-degree of a vertex in a pseudograph is the number of pseudo-edges that have v as its endpoint, with self-loops being counted once. As before, we can define the set of vertices that are adjacent to a given vertex. The *reduced neighborhood* of v is the function $N_v^R : \mathcal{G}_P \to \mathcal{Q}$, with for each $g_P \in \mathcal{G}_P$,

$$N_v^R(g_P) := \{w \in V(g_P) \setminus \{v\} \mid \exists\, e \in E_P(g_P) \text{ such that } v \text{ and } w \text{ are its endpoints}\}.$$

The *reduced pseudo-degree* of v is then the random variable $D_v^R : \mathcal{G}_P \to \mathbb{N}_0$ defined by:

$$\forall g_P \in \mathcal{G}_P : \quad D_v^R(g_P) := |N_v^R(g_P)|,$$

i.e., the reduced pseudo-degree of a vertex in a pseudograph is the number of vertices adjacent to it. The difference between the pseudo-degree of a vertex and its reduced pseudo-degree is that the latter discards multiple edges and self-loops. Hence, the reduced pseudo-degree and the pseudo-degree of a vertex in a pseudograph coincide if the vertex does not have multiple edges or self-loops. In particular, in a simple pseudograph, the pseudo-degree (and thus the reduced pseudo-degree) of a vertex is equal to its degree. Finally, note that the pseudo-degree and the reduced pseudo-degree of $v \in \mathcal{V}$ in a pseudograph g_P are equal to zero if $v \notin V(g_P)$.

We now define the pseudo-degree profile and the reduced pseudo-degree profile of a vertex in a pseudograph. For each $v \in \mathcal{V}$, the *reduced pseudo-degree profile* of v is the function $K_v^R : \mathcal{G}_P \to \Omega_K$ which can be defined in a way analogous to the neighbor degree profile K_v of v in a network, with the reduced neighborhood taking the role of the neighborhood of a vertex and the reduced pseudo-degree taking the role of the degree of a vertex. The definition of the pseudo-degree profile of a vertex is somewhat more involved, as we have to account for multiple edges and self-loops. First, recall the definitions of Ω_K and Ω_K^t, $t \in \mathbb{N}_0$. For $v \in \mathcal{V}$ and $g_P \in \mathcal{G}_P$, define

$$E_v^P(g_P) := \{e \in E_P(g_P) \mid v \text{ is an endpoint of } e\}$$

to be the set of pseudo-edges of which v is an endpoint in g_P. Also, for each pseudo-edge $e \in E_v^P(g_P)$, define $n_v(e; g_P)$ in the following way. If there is a vertex $w \in V(g_P), w \neq v$, that is an endpoint of e, then $n_v(e; g_P) = w$. Otherwise, e is a self-loop with both endpoints equal to v, and we set $n_v(e; g_P) = v$. Then, for $g_P \in \mathcal{G}_P$ and $v \in \mathcal{V}$ such that $D_v^P(g_P) = 0$, we set $K_v^P(g_P) := 0$. Otherwise, define

$$\begin{aligned}
N_1 &:= E_v^P(g_P), \\
j(1) &:= e, \text{ for some } e \in N_1 \text{ such that} \\
&\qquad \nexists\, e' \in N_1 : D^P_{n_v(e';g_P)}(g_P) > D^P_{n_v(e;g_P)}(g_P), \\
K^P_{v,1}(g_P) &:= D^P_{n_v(j(1);g_P)}(g_P),
\end{aligned}$$

and for $\ell = 2, \ldots, D_v^P(g_P)$:

$$\begin{aligned}
N_\ell &:= N_{\ell-1} \setminus \{j(\ell-1)\}, \\
j(\ell) &:= e, \text{ for some } e \in N_\ell \text{ such that} \\
&\qquad \nexists\, e' \in N_\ell : D^P_{n_v(e';g_P)}(g_P) > D^P_{n_v(e;g_P)}(g_P), \\
K^P_{v,\ell}(g_P) &:= D^P_{n_v(j(\ell);g_P)}(g_P).
\end{aligned}$$

Then, $K_v^P(g_P) := (K^P_{v,1}(g_P), \ldots, K^P_{v,D_v^P(g_P)}(g_P))$ is the *pseudo-degree profile of v in g_P*, and the function $K_v^P : \mathcal{G}_P \to \Omega_K$ is the *pseudo-degree profile of v*.

Example 2.A.1. Consider the pseudograph g_P in Figure 2.6(b). The pseudo-degree of vertex 1 and 3 in g_P is $D_1^P(g_P) = D_3^P(g_P) = 4$, and the pseudo-degree of vertex 2 in g_P is $D_2^P(g_P) = 3$. The pseudo-degree of vertex 4 in g_P is $D_4^P(g_P) = 2$. The reduced pseudo-degree in g_P is 2 for all vertices. Hence,

$$
\begin{aligned}
&K_1^P(g_P) = (4,4,4,3),\\
&K_2^P(g_P) = (4,3,2),\\
&K_3^P(g_P) = (4,4,4,2),\\
&K_4^P(g_P) = (4,3),
\end{aligned}
$$

and the reduced pseudo-degree profile in g_P is $(2,2)$ for all vertices. ◁

Finally, for $t \in \mathbb{N}$ and $\theta = (\theta_1, \ldots, \theta_t) \in \Omega_K^t$, define

$$\|\theta\| = \max\{t, \theta_1, \ldots, \theta_t\}.$$

Theorem 2.3.9 uses Lemma 2.A.3, which in turn uses Lemma 2.A.2.

Lemma 2.A.2. *Suppose $X_1, X_2, \ldots$ are independent and identically distributed random variables on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with finite kth moment, where $k \in \mathbb{N}$. Then, as $n \to \infty$,*

$$\frac{\max_{\ell\in\{1,\ldots,n\}} X_\ell}{n^{1/k}} \xrightarrow{\mathrm{P}} 0 \qquad \textit{and} \qquad \mathbb{E}\left[\frac{\max_{\ell\in\{1,\ldots,n\}} X_\ell^k}{n}\right] \to 0.$$

Proof. We start by proving the first claim. Let $\varepsilon > 0$. Then,

$$
\begin{aligned}
\mathbb{P}\left(\max_{\ell\in\{1,\ldots,n\}} X_\ell \le \varepsilon n^{1/k}\right) &= \mathbb{P}\left(\max_{\ell\in\{1,\ldots,n\}} X_\ell^k \le \varepsilon^k n\right)\\
&= \left(1 - \mathbb{P}\left(X_1^k > \varepsilon^k n\right)\right)^n.
\end{aligned}
$$

By Lemma 2.2.35, we thus have

$$\mathbb{P}\left(\max_{\ell\in\{1,\ldots,n\}} X_\ell \le \varepsilon n^{1/k}\right) \ge \left(1 - \frac{1}{\varepsilon^k n}\mathbb{E}\left[X_1^k \mathbf{1}_{[X_1^k \ge \varepsilon^k n]}\right]\right)^n.$$

Since $1 - x \ge e^{-2x}$ for $x \ge 0$ sufficiently small, it holds that for n sufficiently large,

$$\mathbb{P}\left(\max_{\ell\in\{1,\ldots,n\}} X_\ell \le \varepsilon n^{1/k}\right) \ge e^{\frac{2}{\varepsilon^k}\mathbb{E}\left[X_1^k \mathbf{1}_{[X_1^k \ge \varepsilon^k n]}\right]}.$$

Since

$$\mathbb{E}\left[X_1^k \mathbf{1}_{[X_1^k \ge \varepsilon^k n]}\right] \to 0$$

as $n \to \infty$ when X_1 has finite kth moment,

$$\mathbb{P}\left(\max_{\ell \in \{1,\dots,n\}} X_\ell \leq \varepsilon n^{1/k}\right) \to 1$$

as $n \to \infty$, which proves the first claim. To prove the second claim, notice that for all $n \in \mathbb{N}$,

$$\max_{\ell \in \{1,\dots,n\}} X_\ell^k \leq \sum_{\ell=1}^{n} X_\ell^k,$$

so that the second claim follows by dominated convergence. □

Lemma 2.A.3. *For each $n \in \mathbb{N}$, let the number of half-edges of a vertex be distributed as ξ, with support in $\mathbb{N}$, and suppose it has finite first moment. Let $(\mathcal{G}_P^{(n)}, \mathscr{F}_P^{(n)}, \mathbb{P}_{\xi,P}^{(n)})$ be defined as above. Then, for each fixed $\theta \in \Omega_K$ such that $\mathbb{P}_{\xi,P}^{(n)}(K_1^P = \theta) > 0$ for infinitely many n,*

$$\lim_{n\to\infty} \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P \neq K_1^R \mid K_1^P = \theta\right) = 0.$$

Proof. Note that since for each $n \in \mathbb{N}$, we condition on the event $\{g_P \in \mathcal{G}_P^{(n)} \mid K_1^P(g_P) = \theta\}$ for fixed $\theta \in \Omega_K$, the pseudo-degrees of vertex 1 and his neighbors are bounded by $\|\theta\|$. Let $n \in \mathbb{N}$ such that $\mathbb{P}_{\xi,P}^{(n)}(K_1^P = \theta) > 0$. For each vertex $v \in V^{(n)}$, number the half-edges of v arbitrarily in each realization. For each $i, j \in V^{(n)}$, let $\mathbf{1}_{st}^{ij}$ be the indicator function of the event that half-edge s of vertex i is connected to half-edge t of vertex j. Note that we allow for $i = j$. We define the following random variables:

$$S = \sum_{1 \leq s < t \leq D_1^P} \mathbf{1}_{st}^{11},$$

$$\widehat{S} = \sum_{i=2}^{n} \sum_{s=1}^{D_1^P} \sum_{t=1}^{D_i^P} \sum_{\substack{1 \leq u < v \leq D_i^P,\\ u \neq t, v \neq t}} \mathbf{1}_{st}^{1i} \mathbf{1}_{uv}^{ii},$$

$$M = \sum_{i=1}^{n} \sum_{1 \leq s < t \leq D_1^P} \sum_{1 \leq u \neq v \leq D_i^P} \mathbf{1}_{su}^{1i} \mathbf{1}_{tv}^{1i},$$

$$\widehat{M} = \sum_{2 \leq i \neq j \leq n} \sum_{s=1}^{D_1^P} \sum_{\substack{1 \leq t_1 \neq t_2 \neq t_3 \leq D_i^P,\\ t_1 \neq t_3}} \sum_{1 \leq u_1 < u_2 \leq D_j^P} \mathbf{1}_{st_1}^{1i} \mathbf{1}_{t_2 u_1}^{ij} \mathbf{1}_{t_3 u_2}^{ij}.$$

That is, the random variable S counts the number of self-loops of vertex 1, while $\widehat{S}$ counts the number of self-loops of the neighbors of 1. Similarly, M equals the number of multiple edges of vertex 1, while $\widehat{M}$ is the number of multiple edges of the neighbors of 1 with other vertices. Notice that the number of summands is random.

Then, clearly,

$$\{g_P \in \mathcal{G}_P^{(n)} \mid K_1^R(g_P) = K_1^P(g_P)\} = \{g_P \in \mathcal{G}_P^{(n)} \mid S(g_P) + \widehat{S}(g_P) + M(g_P) + \widehat{M}(g_P) = 0\},$$

i.e., the event that the reduced pseudo-degree profile of vertex 1 coincides with its pseudo-degree profile is equal to the event that the sum of the self-loops of vertex 1 and his neighbors and the multiple edges of vertex 1 and his neighbors are zero. Hence, it is sufficient to show that $S, \widehat{S}, M, \widehat{M}$ all converge to zero in probability (given that $K_1^P = \theta$), i.e., it suffices to show that

$$\forall \varepsilon > 0 : \quad \lim_{n\to\infty} \mathbb{P}_{\xi,P}^{(n)}(X \geq \varepsilon \mid K_1^P = \theta) = 0$$

for $X = S, \widehat{S}, M, \widehat{M}$. By Markov's inequality, it is thus sufficient to show that

$$\lim_{n\to\infty} \mathbb{E}_{\xi,P}^{(n)}\left[X \;\middle|\; K_1^P = \theta\right] = 0,$$

where $\mathbb{E}_{\xi,P}^{(n)}$ is the expectation operator associated with the probability measure $\mathbb{P}_{\xi,P}^{(n)}$. Then,

$$\begin{aligned}
\mathbb{E}_{\xi,P}^{(n)}\left[S \;\middle|\; K_1^P = \theta\right] &= \mathbb{E}_{\xi,P}^{(n)}\left[\frac{1}{2}\left(\frac{D_1^P(D_1^P - 1)}{\sum_{j=1}^n D_j^P - 1}\right) \;\middle|\; K_1^P = \theta\right] \\
&\leq \frac{1}{2}\,\mathbb{E}_{\xi,P}^{(n)}\left[\left(D_1^P\right)^2 \;\middle|\; K_1^P = \theta\right]\mathbb{E}_{\xi,P}^{(n)}\left[\frac{1}{\sum_{j=2}^n D_j^P - 1}\right] \\
&\leq \frac{1}{2}\,\|\theta\|^2\,\mathbb{E}_{\xi,P}^{(n)}\left[\frac{1}{\sum_{j=2}^n D_j^P - 1}\right]
\end{aligned}$$

where in the last line we have used that the pseudo-degrees of vertex 1 and his neighbors are bounded by $\|\theta\|$ (conditional on the event $[K_1^P = \theta]$). Write

$$\mathbb{E}_{\xi,P}^{(n)}\left[\frac{1}{\sum_{j=2}^n D_j^P - 1}\right] = \frac{1}{n}\,\mathbb{E}_{\xi,P}^{(n)}\left[\frac{n}{\sum_{j=2}^n D_j^P - 1}\right].$$

By the strong law of large numbers, as $n \to \infty$,

$$\frac{1}{n}\sum_{j=2}^n D_j^P \overset{\text{a.s.}}{\to} \sum_{t\in\mathbb{N}_0} t\xi(t),$$

where we have used that the number of half-edges of each vertex (distributed as ξ) has finite mean.[12] Note that $\sum_{t\in\mathbb{N}_0} t\xi(t) > 0$ as $\xi(0) = 0$ by assumption. Hence, as $n \to \infty$,

$$\frac{n}{\sum_{j=2}^{n} D_j - 1} \xrightarrow{\text{a.s.}} \frac{1}{\sum_{t\in\mathbb{N}_0} t\xi(t)}.$$

Furthermore, note that

$$\frac{n}{\sum_{j=2}^{n} D_j - 1} \leq \frac{n}{n-1},$$

so that $n/(\sum_{j=2}^{n} D_j - 1)$ is bounded by 2 for $n \geq 2$. Therefore, by bounded convergence,

$$\mathbb{E}_{\xi,P}^{(n)}\left[S \mid K_1^P = \theta\right] \to 0$$

as $n \to \infty$. Similarly, for $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E}_{\xi,P}^{(n)}\left[\widehat{S} \mid K_1^P = \theta\right] &= \mathbb{E}_{\xi,P}^{(n)}\left[D_1^P \sum_{i=2}^{n} \frac{\frac{1}{2} D_i^P (D_i^P - 1)(D_i^P - 2)}{\left(\sum_{j=1}^{n} D_j^P - 1\right)\left(\sum_{j=1}^{n} D_j^P - 3\right)} \;\middle|\; K_1^P = \theta\right] \\
&\leq \frac{1}{2} \|\theta\|^4 \, \mathbb{E}_{\xi,P}^{(n)}\left[\frac{n-1}{\left(\sum_{j=1}^{n} D_j^P - 1\right)\left(\sum_{j=1}^{n} D_j^P - 3\right)}\right] \\
&\leq \frac{1}{2} \|\theta\|^4 \, \mathbb{E}_{\xi,P}^{(n)}\left[\frac{1}{\sum_{j=1}^{n} D_j^P - 3}\right].
\end{aligned}$$

Again, by the strong law of large numbers and bounded convergence,

$$\mathbb{E}_{\xi,P}^{(n)}\left[\widehat{S} \mid K_1^P = \theta\right] \to 0.$$

[12] Strictly speaking, the random variables need to be defined on the same probability space in order for the strong law of large numbers to hold. We can do this by defining an underlying probability space that contains all relevant random variables (e.g. Billingsley, 1999, also see Remark 2.3.3 and Ch. 5).

Also,

$$
\begin{aligned}
&\mathbb{E}^{(n)}_{\xi,P}\left[M \,\middle|\, K_1^P = \theta\right] \\
&\quad = \mathbb{E}^{(n)}_{\xi,P}\left[\frac{1}{2} D_1^P \left(D_1^P - 1\right) \sum_{i=1}^{n} \frac{D_i^P (D_i^P - 1)}{\left(\sum_{j=1}^n D_j^P - 1\right)\left(\sum_{j=1}^n D_j^P - 3\right)} \,\middle|\, K_1^P = \theta\right] \\
&\quad \le \frac{1}{2} \|\theta\|^4 \, \mathbb{E}^{(n)}_{\xi,P}\left[\frac{n-1}{\left(\sum_{j=1}^n D_j^P - 1\right)\left(\sum_{j=1}^n D_j^P - 3\right)}\right] \\
&\quad \le \frac{1}{2} \|\theta\|^4 \, \mathbb{E}^{(n)}_{\xi,P}\left[\frac{1}{\sum_{j=1}^n D_j^P - 3}\right],
\end{aligned}
$$

so that, by the strong law of large numbers and bounded convergence,

$$\mathbb{E}^{(n)}_{\xi,P}\left[M \,\middle|\, K_1^P = \theta\right] \to 0.$$

Finally,

$$
\begin{aligned}
&\mathbb{E}^{(n)}_{\xi,P}\left[\widehat{M} \,\middle|\, K_1^P = \theta\right] \\
&\quad = \mathbb{E}^{(n)}_{\xi,P}\left[\frac{1}{2} D_1^P \sum_{2 \le i \ne j \le n} \frac{D_i^P \left(D_i^P - 1\right)\left(D_i^P - 2\right) D_j^P \left(D_j^P - 1\right)}{\left(\sum_{\ell=1}^n D_\ell^P - 1\right)\left(\sum_{\ell=1}^n D_\ell^P - 3\right)\left(\sum_{\ell=1}^n D_\ell^P - 5\right)} \,\middle|\, K_1^P = \theta\right] \\
&\quad \le \frac{1}{2} \|\theta\|^4 \, \mathbb{E}^{(n)}_{\xi,P}\left[\frac{(n-1) \sum_{j=3}^n D_j^P \left(D_j^P - 1\right)}{\left(\sum_{\ell=1}^n D_\ell^P - 1\right)\left(\sum_{\ell=1}^n D_\ell^P - 3\right)\left(\sum_{\ell=1}^n D_\ell^P - 5\right)}\right] \\
&\quad \le \frac{1}{2} \frac{n-1}{n-3} \|\theta\|^4 \, \mathbb{E}^{(n)}_{\xi,P}\left[\frac{\max_{j \in \{3,4,\ldots,n\}} D_j^P}{\sum_{\ell=1}^n D_\ell^P - 5}\right] \\
&\quad = \frac{1}{2} \frac{n-1}{n-3} \|\theta\|^4 \, \mathbb{E}^{(n)}_{\xi,P}\left[\frac{\max_{j \in \{3,4,\ldots,n\}} D_j^P}{n-2} \frac{n-2}{\sum_{\ell=1}^n D_\ell^P - 5}\right] \\
&\quad \le \frac{1}{2} \frac{n-1}{n-3} \frac{n-2}{n-5} \|\theta\|^4 \, \mathbb{E}^{(n)}_{\xi,P}\left[\frac{\max_{j \in \{3,4,\ldots,n\}} D_j^P}{n-2}\right].
\end{aligned}
$$

Note that for $n \ge 8$,

$$\frac{n-1}{n-3} \frac{n-2}{n-5} \le 4.$$

Hence, recalling Remark 2.3.3 (cf. footnote 12), by Lemma 2.A.2,

$$\mathbb{E}^{(n)}_{\xi,P}\left[\widehat{M} \,\middle|\, K_1^P = \theta\right] \to 0. \qquad \square$$

For ease of notation, for each $t \in \mathbb{N}$ and each $\theta \in \Omega_K^t$, define:

$$f(\theta) := \xi(t)\, M(\theta) \prod_{\substack{s\in\mathbb{N}:\\ \xi(s)>0}} \left(\frac{s\xi(s)}{\sum_{r\in\mathbb{N}} r\xi(r)} \right)^{c_s(\theta)}.$$

We can now prove Theorem 2.3.9. First, note that the distribution of D_j^P of each vertex j is very close to ξ, as the number of half-edges of each vertex is distributed as ξ, and each half-edge becomes a pseudo-edge in the corresponding pseudograph, unless the sum of all half-edges is odd and the vertex is selected to have its number of half-edges increased by 1. When $n \to \infty$, the probability that the pseudo-degree of vertex 1 or of one of its neighbors is increased goes to zero,[13] so that for each $t \in \mathbb{N}$ and each $\theta \in \Omega_K^t$,

$$\mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta\right) \to f(\theta)$$

as $n \to \infty$. Let $t \in \mathbb{N}$ and $\theta \in \Omega_K^t$. Then, for each $\varepsilon > 0$, there exists $T_\varepsilon \in \mathbb{N}$ such that

$$\sum_{t=1}^{T_\varepsilon} \sum_{\theta:||\theta||=t} f(\theta) \geq 1 - \varepsilon.$$

Since $\Theta_{T_\varepsilon} := \{\theta \in \Omega_K \mid ||\theta|| \leq T_\varepsilon\}$ is a finite set, $\mathbb{P}_{\xi,P}^{(n)}(K_1^P = \theta)$ converges to $f(\theta)$ uniformly for those θ such that $||\theta|| \leq T_\varepsilon$. In particular, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n > N_\varepsilon$,

$$\sum_{t\in\Theta_{T_\varepsilon}} \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta\right) \geq 1 - 2\varepsilon. \tag{2.8}$$

Let $n \in \mathbb{N}$ be such that $\mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta\right) > 0$. Define

$$\Omega_K^{(n)} := \left\{\theta' \in \Omega_K \,\middle|\, \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta'\right) > 0\right\}.$$

13 Note that it is not an event in $\mathscr{F}_P^{(n)}$ that the pseudo-degree of vertex 1 or one of its neighbors is increased. However, we can define an underlying probability space for the random network model with an appropriate σ-field such that it is measurable with respect to that σ-field (also see Remark 2.3.3 Billingsley, 1999).

We have

$$
\begin{aligned}
\mathbb{P}_{\xi}^{(n)}(K_1 = \theta) &= \mathbb{P}_{\xi,P}^{(n)}\left(K_1^R = \theta\right) \\
&= \sum_{\theta' \in \Omega_K^{(n)}} \mathbb{P}_{\xi,P}^{(n)}\left(K_1^R = \theta \mid K_1^P = \theta'\right) \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta'\right) \\
&= \sum_{\theta' \in \Omega_K^{(n)}} \Big[\mathbb{P}_{\xi,P}^{(n)}\left(K_1^R = \theta, K_1^P = K_1^R \mid K_1^P = \theta'\right) \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta'\right) + \\
&\qquad \mathbb{P}_{\xi,P}^{(n)}\left(K_1^R = \theta, K_1^P \neq K_1^R \mid K_1^P = \theta'\right) \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta'\right)\Big] \\
&= \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = K_1^R \mid K_1^P = \theta\right) \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta\right) + \\
&\qquad \sum_{\theta' \in \Omega_K^{(n)}} \mathbb{P}_{\xi,P}^{(n)}\left(K_1^R = \theta, K_1^P \neq K_1^R \mid K_1^P = \theta'\right) \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta'\right). \quad (2.9)
\end{aligned}
$$

Consider the second term of (2.9). We have

$$
\begin{aligned}
\sum_{\theta' \in \Omega_K^{(n)}} \mathbb{P}_{\xi,P}^{(n)}\left(K_1^R = \theta, K_1^P \neq K_1^R \mid K_1^P = \theta'\right) \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta'\right) \leq \sum_{\theta' \in \Omega_K^{(n)} \setminus \Theta_{T_\varepsilon}} \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta'\right) \\
+ \sum_{\theta' \in \Theta_{T_\varepsilon} \cap \Omega_K^{(n)}} \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P \neq K_1^R \,\middle|\, K_1^P = \theta'\right) \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta'\right). \quad (2.10)
\end{aligned}
$$

By (2.8), the first term of (2.10) is at most 2ε for $n > N_\varepsilon$. Consider the second term of (2.10). Using Lemma 2.A.3, we find that

$$
\mathbb{P}_{\xi,P}^{(n)}\left(K_1^P \neq K_1^R \,\middle|\, K_1^P = \theta'\right) \leq \varepsilon
$$

uniformly for $\theta' \in \Theta_{T_\varepsilon}$ for n sufficiently large (since Θ_{T_ε} is a finite set). Hence, for n sufficiently large,

$$
\sum_{\theta' \in \Omega_K^{(n)}} \mathbb{P}_{\xi,P}^{(n)}\left(K_1^R = \theta, K_1^P \neq K_1^R \mid K_1^P = \theta'\right) \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta'\right) \leq 2\varepsilon + \varepsilon. \quad (2.11)
$$

Applying Lemma 2.A.3 to the first term of (2.9), and using (2.11), we obtain

$$
\lim_{n\to\infty} \mathbb{P}_{\xi,P}^{(n)}\left(K_1^R = \theta\right) = \lim_{n\to\infty} \mathbb{P}_{\xi,P}^{(n)}\left(K_1^P = \theta\right) = f(\theta). \qquad \square
$$

Part I

Networks and Games

3 Convergence of beliefs in Bayesian network games

Summary

In this chapter, which is based on Kets (2007b), we study a setting in which players are located on a network and play a fixed game with their neighbors. Players have incomplete information on the network structure. They have a common prior over the network, and in addition, they know the number of connections they have. We study the sensitivity of game-theoretic predictions to the specification of players' beliefs. We show that two priors are close in a strategic sense if and only if they assign similar probabilities to all local events, i.e., to all events involving the types of a player and his neighbors. This means that in order to fully explore the range of possible strategic outcomes, it suffices to vary the type distribution and the correlation among player types.

3.1 Introduction

In many contexts, an agent's well-being primarily depends on the behavior of those with whom he has a direct relationship, rather than on the behavior of the population at large. Indeed, Goolsbee and Klenow (2002) and Tucker (2006) find that an individual's decision to adopt a particular communication technology is primarily influenced by the adoption decisions of those with whom he interacts directly, rather than by the overall adoption level in the population. Also, an agent's connections provide access to various resources such as information, knowledge and capital. For instance, a key success factor for a firm in a high-tech sector such as the biotechnology industry is its position in a network of R&D partnerships (Powell et al., 1996).[1] Hence, in a variety of settings, the networks formed by agents' relations are important in determining economic outcomes. These networks are generally large and complex, and evolve rapidly over time (e.g. Powell et al., 2005). This suggests that agents often do not know the exact structure of the network they belong to.[2] At the same time, it is unclear what beliefs agents have about

1 Other empirical studies that highlight the role of networks in economic settings include Coleman et al. (1966) and Conley and Udry (2005) on the diffusion of new technologies in medicine and agriculture, respectively, Granovetter (1974) on job search, and Fafchamps and Lund (2003) on informal insurance networks in developing countries.

2 Krackhardt and Hanson (1993) report that informal networks are mostly unobservable to senior executives. Also, Powell et al. (1996, p.120) observe that in R&D collaborations in biotechnology, "beneath most formal ties [. . .] lies a sea of informal relations".

their networks.[3] Hence, in settings where agents interact strategically with their neighbors on a network under incomplete information on the network structure, it is important to assess how game-theoretic predictions depend on the assumptions on players' beliefs. This is the topic of the current chapter.

More specifically, we study a setting in which players are located on a network and play a fixed game with their neighbors. Payoffs only depend on a player's own action and characteristics and on the actions and characteristics of his neighbors. Players have a (common) prior over the network, and, in addition, they have some local information. Each player is informed of the number of neighbors he has in the network, i.e., a player's type is his *degree*. This defines a *Bayesian network game*. Since the interest in such games is usually on the effect of network characteristics on the behavior of players, we focus on symmetric equilibria, as in much of the literature (cf. Galeotti et al., 2006; Jackson and Yariv, 2007; Sundararajan, 2005). We define a function that for any two priors gives their *strategic distance*. Loosely speaking, the strategic distance between two priors is small if for any game in which players hold one of these priors, for any symmetric Bayesian-Nash equilibrium in that game, there is a symmetric approximate equilibrium in the associated game with the other prior such that ex ante expected payoffs are close under both equilibria (cf. Kajii and Morris, 1998). If that is the case, players can obtain approximately the same ex ante expected payoffs under both priors, and we say that the two priors are close in a strategic sense. We study the necessary and sufficient conditions for two priors to be close in a strategic sense. We thus consider a type of lower hemicontinuity of the correspondence of (interim) approximate equilibria in Bayesian network games (see Engl, 1995, for a discussion of different continuity concepts).

Our main result (Theorem 3.4.2) shows that two priors are close in a strategic sense if and only if they assign similar probabilities to local events, i.e., events that involve the types of a player and his neighbors. This result has two important implications. Firstly, it indicates that in order to fully explore the possible strategic outcomes in Bayesian network games, it is sufficient to vary the type distribution and the correlation among player types. Hence, on the one hand, varying the type distribution, as has been the focus of much of the literature so far (see below for a discussion of this literature), is not enough. On the other hand, the result limits the set of priors that one needs to consider, as we show that priors need only be varied along two dimensions.

[3] Evidence suggests that agents use simple heuristics (Janicik and Larrick, 2005), and that their perception of the network is biased (e.g. Kumbasar et al., 1994), even in an environment with strong incentives (Johnson and Orbach, 2002).

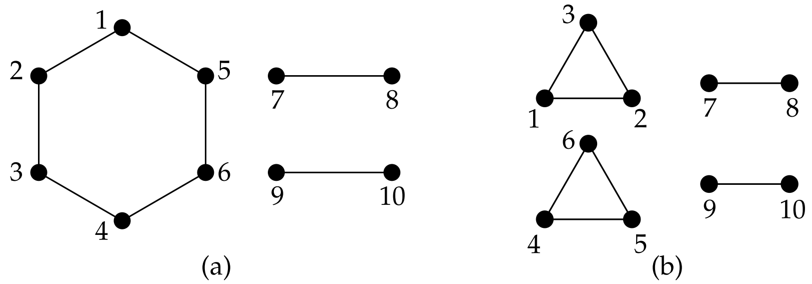


Figure 3.1. The networks in (a) and (b) are identical in their local properties. That is, in both networks, there are 6 vertices with degree 2 that are only connected to other vertices with degree 2, and 4 vertices with degree 1, which are connected exclusively with other vertices with degree 1.

Secondly, Theorem 3.4.2 implies that we can interpret a Bayesian network game as a set of overlapping "local" games, so that we do not need to concern ourselves with the nonlocal features of priors. This can best be understood by means of a concrete example. For instance, consider two priors, and suppose that one of the priors assigns positive probability only to networks that are isomorphic to the network in Figure 3.1(a), with each of the networks in this isomorphism class having equal probability, while the other prior assigns positive probability only to networks isomorphic to the network in Figure 3.1(b), and each of the networks in this isomorphism class has equal probability. Clearly, these priors are identical in terms of the probabilities assigned to local events, i.e., in terms of the events involving the types of a player and his direct neighbors, but very different in terms of the probabilities they assign to global events, i.e., to different networks. We show that, despite the differences on the global level, the two priors are identical in terms of their strategic implications.

The motivation for the question we study comes from empirical work. The last few years, there has been a surge in empirical work on networks, owing to the availability of data on large-scale networks such as the World Wide Web (see Jackson, 2008, for an overview). This work has shown that networks that are relevant for economic applications are characterized by a number of properties. Some of these properties relate to the local environment of a player. For instance, an important property of networks is the distribution of the number of direct contacts that people have. Other properties are defined on a larger scale. The clustering coefficient of a network, for instance, quantifies the extent to which friends of your friends are also your friends. Another example is the degree correlation, i.e., the correlation in the number of contacts people have. An important question for game-theoretic applications is then how these different properties affect strategic interactions on networks.

So far, most of the literature has focused on the effect of varying the degree distribution, i.e., the distribution of player types, on game-theoretic outcomes, assuming that players' types are independent (e.g. Galeotti and Vega-Redondo, 2005; Jackson and Yariv, 2007; López-Pintado, 2006; Sundararajan, 2005), using the random network model with a given degree distribution discussed in Section 2.3.2.[4] An important question is whether game-theoretic predictions obtained under the assumption that players' types are independent continue to hold if we relax this assumption. The current chapter shows that this is not the case. We show that for two priors to give rise to similar outcomes (from a player's ex ante perspective), it is not sufficient that they are close in terms of the type distribution they induce, they also need to be close in terms of the correlation among player types. Hence, while varying the type distribution may be a good starting point, the current chapter shows that one needs to go beyond the class of random network models with a given degree distribution to fully explore the range of strategic outcomes. At the same time, our result restricts the set of priors that one needs to consider, as we show that priors need only be varied along two dimensions, the distribution of types and the correlation among player types.

To illustrate these points, we present a simple example in Section 3.4.2. We study a game in which players can choose whether to invest or not. Not investing gives a payoff of zero, independent of others' actions, while investing is only profitable if all neighbors invest. Hence, this is a game of strategic complements. We compare two priors which are identical in terms of the type distribution they induce, but which differ in terms of the correlation among types. We show that there exists a symmetric strategy profile that is a Bayesian-Nash equilibrium under one prior that is not an (approximate) equilibrium under the other prior, and vice versa, and ex ante expected payoffs under equilibria under the two priors differ. These priors are thus different in terms of strategic predictions.

The work in this chapter is related to two distinct literatures. Firstly, the current chapter contributes to the literature on Bayesian network games (e.g. Galeotti et al., 2006; Galeotti and Vega-Redondo, 2005; Jackson and Yariv, 2007; Sundararajan, 2005). This literature studies the effect of network structure on game-theoretic outcomes. In particular, Galeotti et al. (2006) study the effect of varying the type

[4] Note however, that degrees are only *asymptotically* independent in this model (Proposition 2.3.9).

distribution and the correlation among players' types in a particular way in games with strategic complements and substitutes. They show that predictions change when the type distribution and the correlation among players' types are varied. This illustrates that it is important to go beyond the assumption of independent types made in the earlier literature. The current chapter complements the work of Galeotti et al. (2006) in two ways. First, we show that varying the type distribution and the type correlation, as Galeotti et al. (2006) do, is indeed sufficient to capture all possible strategic behavior in any class of Bayesian network games. Second, while Galeotti et al. (2006) focus on gradual changes in equilibrium behavior as priors are continuously varied in terms of the distribution of types and the type correlation, our results emphasize that it is possible to obtain qualitatively different outcomes if priors differ in these two dimensions (see e.g. the example in Section 3.4.2).

The second literature to which this chapter is related is the literature on (payoff) continuity in games. Continuity issues in general Bayesian games have been studied by a number of authors (Kajii and Morris, 1998; Milgrom and Weber, 1985; Monderer and Samet, 1996). The question we study is similar to the question studied by Kajii and Morris (1998). While Kajii and Morris (1998) study payoff continuity in general Bayesian games, we restrict attention to the class of Bayesian network games. Moreover, we focus on symmetric equilibria. By exploiting the symmetry of the game, we are able to weaken the conditions of Kajii and Morris (1998). That this can be done is not obvious. While payoffs only depend directly on the actions of neighbors in our setting, actions and beliefs of those further away in the network may have a considerable effect on the payoffs to a player, through the effect on the neighbors of those players and the neighbors of the neighbors of those players, and so on. There is thus a tension between the local nature of the payoffs and the interdependencies intrinsic to the network setting. Our results show that Bayesian network games can nevertheless be treated as a collection of overlapping local games. The value of this result is that it implies that priors need only be varied in terms of the type distribution and the correlation among player types that they induce to explore the possible strategic outcomes.

This chapter is organized as follows. Preliminaries are discussed in Section 3.2. Bayesian network games are defined in Section 3.3. The main result is presented in Section 3.4. Section 3.5 concludes. Proofs that are not included in the main text can be found in Appendix 3.A.

3.2 Preliminaries

In our framework, players are located on a network. Networks and random network models have been defined in Section 2.3.1; for ease of reference, we briefly recall the most important definitions here. A *network* g is a pair consisting of a finite, nonempty set $V(g)$ of *vertices* and a finite set $E(g)$ of *edges*, with an edge being an unordered pair of two distinct vertices. Let g be a network. If $\{i, j\} \in E(g)$, where $i, j \in V(g), i \neq j$, then i and j are *neighbors* in g; alternatively, we say that i and j are *adjacent* in g. For notational simplicity, an edge $\{i, j\} \in E(g)$ is sometimes denoted by ij.

A *random network model* is a probability space (i.e., a triple consisting of a sample space, a σ-algebra, and a probability measure on the σ-algebra) in which the sample space is a nonempty (finite or countable) set of networks. The probability measure on the σ-algebra is determined by a random construction process, the experiment. The outcome of such an experiment is called a *random network*. In the current setting, we associate a player with each vertex, so that edges represent the relations between players. In the following, we therefore refer to players rather than to vertices. Furthermore, random network models represent players' beliefs. Throughout this chapter, we therefore refer to a random network model as a *network belief system*.

Let $n \in \mathbb{N}$ and $V^{(n)} := \{1, \ldots, n\}$. Let $\mathcal{G}^{(n)}$ be the set of all networks with player set $V^{(n)}$, and let $\mathscr{F}^{(n)}$ be the set of all subsets of $\mathcal{G}^{(n)}$. Let $\mathcal{M}^{(n)}$ be the set of all probability measures on $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)})$. Let $i \in V^{(n)}$ and $g \in \mathcal{G}^{(n)}$. The *degree* D_i of player i is a random variable that gives for each $g \in \mathcal{G}^{(n)}$ the number of neighbors of i in g. The *neighborhood* $N_i(g)$ of i in g is the set of neighbors of i in g. The *neighbor degree profile* of i in g is a list of the degrees of the neighbors of player i, in a non-increasing order. For $t \in \mathbb{N}$, Ω_K^t is the set of all neighbor degree profiles of a player with degree t, and we define $\Omega_K := \bigcup_t \Omega_K^t$.

We are interested in the case in which players are ex ante identical in terms of their network position. Throughout this chapter, we therefore make the following assumption on network belief systems:

Assumption 3.A (Exchangeability). Let $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mu)$ be a network belief system. The neighbor degree profiles $K_1, K_2, \ldots, K_n$ are exchangeable. That is, for any $k \in \{1, \ldots, n\}$, $i_1, \ldots, i_k \in V^{(n)}$, the random vector $(K_{i_1}, K_{i_2}, \ldots, K_{i_k})$ has the same distribution as the random vector $(K_{\pi(i_1)}, K_{\pi(i_2)}, \ldots, K_{\pi(i_k)})$ for any permutation

$\pi : \{i_1, \ldots, i_k\} \to \{i_1, \ldots, i_k\}$. In particular, for all $i, j \in V^{(n)}$, for all $\theta \in \{0, \ldots, n-1\}$,

$$\mu(\{g \in \mathcal{G}^{(n)} \mid D_i(g) = \theta\}) = \mu(\{g \in \mathcal{G}^{(n)} \mid D_j(g) = \theta\}),$$

i.e., the probability that a player has a certain degree is the same for each player.

Network belief systems $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mu)$ with this property exist. For instance, let μ be the uniform distribution on the finite set $\mathcal{G}^{(n)}$. Also the network belief systems discussed in Chapter 5 satisfy this property.

Definition 3.2.1. *Let $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mu)$ be a network belief system such that Assumption 3.A is satisfied. Then, the* degree distribution of $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mu)$ *is given by* $\left(p^{(n)}(t)\right)_{t \in \mathbb{N}_0}$*, where*

$$\forall t \in \mathbb{N}_0 : \quad p^{(n)}(t) := \mu\Big(\{g \in \mathcal{G}^{(n)} \mid D_1(g) = t\}\Big).$$

That is, the degree distribution of a network belief system gives for each $t \in \mathbb{N}_0$ the probability that a player selected uniformly at random from the network has degree t.

Finally, for notational convenience, we assume:

Assumption 3.B (No isolated vertices). The network belief system $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mu)$ is such that with probability 1, each player has at least one neighbor. That is,

$$\mu(\{g \in \mathcal{G}^{(n)} \mid D_i(g) > 0 \text{ for all } i \in V(g)\}) = 1. \quad \triangleleft$$

3.3 Bayesian network games

A Bayesian network game is a Bayesian game where the states of nature are networks drawn according to a network belief system and in which each player is informed of the number of neighbors he has. Formally, let $n \in \mathbb{N}$. A *Bayesian network game* is a Bayesian game

$$\langle N, \mathcal{G}^{(n)}, (A_i)_{i \in N}, (T_i)_{i \in N}, (\tau_i)_{i \in N}, \mu, (u_i)_{i \in N} \rangle,$$

where $N = \{1, \ldots, n\}$ is the set of players and $(\mathcal{G}^{(n)}, \mathscr{F}^{(n)}, \mu)$ is a network belief system on vertex set $V^{(n)} = N$ such that Assumptions 3.A and 3.B are satisfied. The probability measure μ is players' *(common) prior*. Each player $i \in N$ has a nonempty, finite set A_i of pure strategies or *actions*. If the state of nature/network is $g \in \mathcal{G}^{(n)}$, player i's private information is his degree. Hence, the set of *types* or signals of player i is $T_i = \{0, \dots, n-1\} =: T$ and his *signal function* $\tau_i : \mathcal{G}^{(n)} \to T$ assigns to each network $g \in \mathcal{G}^{(n)}$ the degree $\tau_i(g) := D_i(g)$ of player i. Finally, each player $i \in N$ has a von Neumann-Morgenstern *utility function* $u_i : (\times_{i \in N} A_i) \times \mathcal{G}^{(n)} \to \mathbb{R}$.

Henceforth, we speak of type and *neighbor type profile* rather than of degree and neighbor degree profile. Also, we will refer to the *type distribution* of a network belief system that satisfies Assumption 3.A to denote its degree distribution (Definition 3.2.1).

We assume that there exists a finite set A such that $A_i = A$ for all $i \in N$. Furthermore, we assume that there exists a profile of *local payoff functions* $v = (v_t)_{t \in T}$ that for each $t \in T$ gives the payoff to a player of type t. More specifically, for $t = 0$, v_t is a real function on A, and for each $(a_i, a_{-i}) \in A^n, g \in \mathcal{G}^{(n)}$ and $i \in V^{(n)}$ such that $\tau_i(g) = 0$, $u_i(a_i, a_{-i}, g) = v_0(a_i)$, i.e., the payoffs to an isolated player only depend on his own type and action. For $t > 0$, v_t is a real function on $A \times A^t \times T^t$ that is symmetric in A^t and T^t, i.e., for all permutations π on $\{1, \dots, t\}$, for all $b \in A, (a_1, \dots, a_t) \in A^t, (\theta_1, \dots, \theta_t) \in T^t$,

$$v_t\Big(b, (a_1, \dots, a_t), (\theta_1, \dots, \theta_t)\Big) = v_t\Big(b, (a_{\pi(1)}, \dots, a_{\pi(t)}), (\theta_{\pi(1)}, \dots, \theta_{\pi(t)})\Big).$$

Then, for each $i \in V^{(n)}$, $g \in \mathcal{G}^{(n)}$ and $a = (a_1, \dots, a_n) \in A^n$,

$$u_i(a, g) = v_{\tau_i(g)}(a_i, (a_j)_{j \in N_i(g)}, (\tau_j(g))_{j \in N_i(g)}).$$

That is, a player's payoff only depends on his own action and type, and the actions and types of his neighbors, and does so in an anonymous way. The *bound* B of a profile of local payoff functions v is defined as:

$$B := \max_{\substack{t \in T \setminus \{0\}; \\ \theta \in T^t; a, a' \in A^{t+1}}} \Big\{ |v_t(a, \theta) - v_t(a', \theta)|, |v_t(a, \theta)| \Big\}.$$

This maximum exists, as the signal set T and the action set A are finite.

Throughout this chapter, we fix the player set N and the action set A. A Bayesian network game is then fully characterized by the common prior μ and its profile v of local payoff functions. We henceforth denote a Bayesian network game $\langle N, \mathcal{G}^{(n)}, (A_i)_{i \in N}, (T_i)_{i \in N}, (\tau_i)_{i \in N}, \mu, (u_i)_{i \in N} \rangle$ by the pair (μ, v).

For $i \in N$, a *(mixed) strategy* for player i is a function $\sigma_i : T \to \Delta(A)$. Denote the set of all strategies by Σ. The probability that action $a_i \in A$ is played under strategy σ_i by player $i \in N$ given that he has type $t_i \in T$ is denoted by $\sigma_i(a_i \mid t_i)$. A *strategy profile* is a function $\sigma = (\sigma_i)_{i \in N} \in \Sigma^n$, with σ_i a strategy of player i for each $i \in N$. For strategy profile $\sigma = (\sigma_j)_{j \in N}$ and $i \in N$, we write σ_{-i} to denote the strategy profile $\sigma = (\sigma_j)_{j \in N \setminus \{i\}}$ of the opponents of i. We say that a strategy profile σ is *symmetric* if $\sigma_i = \sigma_j$ for all $i, j \in N$.

We can now define expected payoffs. First, we introduce some notation that will be useful in the following. For $t \in T$, $F \in \mathscr{F}_K$, and $\theta \in \Omega_K$, define

$$\begin{aligned}
&\mu(t) := \mu(\{g' \in \mathcal{G}^{(n)} \mid D_1(g') = t\}), \\
&\mu(F) := \mu(\{g' \in \mathcal{G}^{(n)} \mid K_1(g') \in F\}), \\
&\mu(\theta) := \mu(\{g' \in \mathcal{G}^{(n)} \mid K_1(g') = \theta\}).
\end{aligned}$$

By Assumption 3.A, $\mu(t)$ is the prior probability that any fixed player has type t, and $\mu(F)$ is the prior probability that the neighbor type profile of any fixed player lies in the set F. Finally, $\mu(\theta)$ is the prior probability that a fixed player has neighbor type profile θ.

We also introduce some short-hand notation for various conditional probabilities. Let $t \in T$ be such that $\mu(t) > 0$. For $g \in \mathcal{G}^{(n)}$, $F \in \mathscr{F}_K$ and $\theta \in \Omega_K$, let

$$\begin{aligned}
&\mu(g \mid t) := \mu(\{g\} \mid \{g' \in \mathcal{G}^{(n)} \mid D_1(g') = t\}), \\
&\mu(F \mid t) := \mu(\{g' \in \mathcal{G}^{(n)} \mid K_1(g') \in F\} \mid \{g' \in \mathcal{G}^{(n)} \mid D_1(g') = t\}), \\
&\mu(\theta \mid t) := \mu(\{g' \in \mathcal{G}^{(n)} \mid K_1(g') = \theta\} \mid \{g' \in \mathcal{G}^{(n)} \mid D_1(g') = t\}).
\end{aligned}$$

In words, $\mu(g \mid t)$ is the conditional probability that the network is g given that player 1 has degree t. Similarly, $\mu(F \mid t)$ is the conditional probability that the neighbor type profile of player 1 lies in the set F given that he has type t. Finally, $\mu(\theta \mid t)$ is the conditional probability that the neighbor type profile of player 1 is equal to θ, given that he has type t.

Then, the *interim expected payoff* to player $i \in N$ of action $a_i \in A$ under common prior $\mu \in \mathcal{M}^{(n)}$ when he receives signal $t_i \in T$ with $\mu(t_i) > 0$ and when the other players play according to the strategy profile $(\sigma_j)_{j \in N \setminus \{i\}} \in \Sigma^{n-1}$ is given by

$$\begin{aligned}
\varphi_i(a_i, \sigma_{-i}; t_i, \mu) &:= \sum_{g \in \mathcal{G}^{(n)}} \mu(g \mid t_i) u_i\Big(a_i, \big(\sigma_j(\tau_j(g))\big)_{j \in N \setminus \{i\}}, g\Big) \\
&= \sum_{g \in \mathcal{G}^{(n)}} \mu(g \mid t_i) v_{t_i}(a_i, \sigma_{N_i(g)}, \tau_{N_i(g)}),
\end{aligned}$$

where we have defined $\sigma_{N_i(g)} := (\sigma_j(\tau_j(g)))_{j \in N_i(g)}$ and $\tau_{N_i(g)} := (\tau_j(g))_{j \in N_i(g)}$. Similarly, the *ex ante expected payoff* to a player $i \in N$ of the strategy profile σ when players' prior is μ is

$$\begin{aligned}\Phi_i(\sigma;\mu) &:= \sum_{g\in\mathcal{G}^{(n)}} \mu(\{g\})u_i(\sigma,g)\\ &= \sum_{\substack{t_i\in T:\\ \mu(t_i)>0}} \mu(t_i)\sum_{a_i\in A}\sigma_i(a_i\mid t_i)\varphi_i(a_i,\sigma_{-i};t_i,\mu).\end{aligned}$$

Definition 3.3.1. *Let $\varepsilon \geq 0$. A strategy profile $\sigma \in \Sigma^n$ is an (interim) ε-*equilibrium *of a Bayesian network game (μ, v) if for each player $i \in N$, for each $t_i \in T$ with $\mu(t_i) > 0$, each $a_i \in A$ with $\sigma_i(a_i \mid t_i) > 0$,*

$$\varphi_i(a_i,\sigma_{-i};t,\mu) \geq \varphi_i(b_i,\sigma_{-i};t,\mu) - \varepsilon$$

for all $b_i \in A$. That is, in an ε-equilibrium, a player can gain at most ε from unilateral deviation. An ε-equilibrium is symmetric *if it is a symmetric strategy profile.*

A Bayesian-Nash equilibrium is a 0-equilibrium. A Bayesian-Nash equilibrium exists for Bayesian network games by standard arguments. We also have the following result:

Proposition 3.3.2. *Let (μ, v) be a Bayesian network game. Then there exists a symmetric Bayesian-Nash equilibrium of (μ, v).*

For a proof, see Appendix 3.A. For $\varepsilon \geq 0$, denote the set of symmetric ε-equilibria of the Bayesian network game (μ, v) by $\mathcal{N}^{\varepsilon}(\mu, v)$. In particular, the set $\mathcal{N}^0(\mu, v)$ denotes the set of symmetric Bayesian-Nash equilibria of (μ, v).

When players play according to a symmetric strategy profile, we can simplify the expressions for players' expected payoffs. A symmetric strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma^n$ can be denoted by $\hat{\sigma} := (\hat{\sigma}_t)_{t\in T}$, with $\hat{\sigma}_t(a) = \sigma_i(a \mid t)$ for any $i \in N$ the probability that a player of type $t \in T$ takes action $a \in A$. Let σ be a symmetric strategy profile, and let $\hat{\sigma} = (\hat{\sigma}_t)_{t\in T}$, with, for all $t \in T$, $\hat{\sigma}_t(\cdot) = \sigma_i(\cdot \mid t)$ for any $i \in N$. For $t \in T$ and type profile $\theta = (\theta_1, \ldots, \theta_t) \in \Omega_K$, we write $\hat{\sigma}_\theta$ to denote $(\hat{\sigma}_{\theta_1}, \ldots, \hat{\sigma}_{\theta_t})$. Then, for $t \in T$ such that $\mu(t) > 0$, and $a \in A$, we define

$$\begin{aligned}\hat{\varphi}_t(a,\hat{\sigma};\mu) &:= \sum_{\theta\in\Omega_K^t} \mu(\theta\mid t)\, v_t(a,\hat{\sigma}_\theta,\theta)\\ &= \varphi_i(a,\sigma_{-i};t,\mu)\quad \text{for any } i\in N\end{aligned}$$

to be the interim expected payoff to an arbitrary player of type t of action a when players play according to the symmetric strategy profile σ and the common prior is μ. Similarly, for a symmetric strategy profile $\sigma \in \Sigma^n$, we define

$$\begin{aligned}\hat{\Phi}(\hat{\sigma};\mu) &:= \sum_{\substack{t\in T:\\ \mu(t)>0}} \mu(t) \sum_{a\in A} \hat{\sigma}_t(a)\,\hat{\varphi}_t(a,\hat{\sigma};\mu)\\ &= \Phi_i(\sigma;\mu) \quad \text{for any } i \in N\end{aligned}$$

to be the ex ante expected payoff to an arbitrary player when players play according to the symmetric strategy profile σ and the prior is μ.

3.4 Strategic convergence

3.4.1 Strategic distance

Our objective is to define a "measure" of similarity of priors such that if two priors are similar according to this measure, then, for each Bayesian network game in which beliefs are given by one of the priors, for each symmetric Bayesian-Nash equilibrium of the game, there exists a symmetric approximate equilibrium in the game with the same profile of local payoff functions but with beliefs given by the other prior such that ex ante payoffs are close under the two equilibria. If that is the case, then, for all possible payoff functions, players can obtain approximately the same payoffs (ex ante) under both priors. In that case, the two priors are similar from players' (ex ante) perspective. At the same time, we do not want to make the conditions on priors to be similar any stricter than necessary. We thus look for the weakest condition that guarantees that the above holds.

Formally, let $\mu, \mu' \in \mathcal{M}^{(n)}$, and let $v = (v_t)_{t\in T}$ be a profile of local payoff functions. For each $\varepsilon \geq 0$, define

$$\chi(\mu,\mu';v,\varepsilon) := \sup_{\sigma\in\mathcal{N}^0(\mu,v)} \inf_{\sigma'\in\mathcal{N}^\varepsilon(\mu',v)} |\hat{\Phi}(\hat{\sigma};\mu) - \hat{\Phi}(\hat{\sigma}';\mu')|,$$

where $\hat{\Phi}$ is the ex ante expected payoff function given profile v of local payoff functions, and σ and σ' are the symmetric strategy profiles corresponding to $\hat{\sigma}$ and $\hat{\sigma}'$, respectively. Hence, $\chi(\mu,\mu';v,\varepsilon)$ is a measure of the difference in outcomes under μ' and μ in terms of ex ante expected payoffs when players play according to a symmetric strategy. More specifically, for a given $\varepsilon \geq 0$, for each symmetric Bayesian-Nash equilibrium under μ, we first fix a symmetric ε-equilibrium under μ' which minimizes the (absolute) difference in ex ante expected payoffs under both equilibria, and we then take a symmetric Bayesian-Nash equilibrium under μ

which maximizes this difference. This formalizes the idea that for *each* symmetric Bayesian-Nash equilibrium of a Bayesian network game with one of the priors, there exists *some* symmetric approximate equilibrium of the Bayesian network game with the other prior, such that ex ante expected payoffs are similar under both equilibria. However, the function $\chi(\mu, \mu'; v, \varepsilon)$ is not symmetric in μ and μ', as we would want. To obtain a symmetric function of μ and μ', let

$$\chi^*(\mu, \mu'; v, \varepsilon) := \max \{\chi(\mu, \mu'; v, \varepsilon), \chi(\mu', \mu; v, \varepsilon)\}.$$

We refer to $\chi^*(\mu, \mu'; v, \varepsilon)$ as the *strategic distance* between μ and μ' for the profile v given ε. The supremum of $\chi^*(\mu, \mu'; v, \varepsilon)$ over v is called the *strategic distance* between μ and μ' given ε.

Note that when ε increases, the set of symmetric ε-equilibria weakly increases, as more and more symmetric strategies will satisfy the equilibrium criterion, and the (absolute) difference in ex ante expected payoffs will decrease weakly. Hence, the interesting case is when ε comes arbitrarily close to 0. This leads us to the following definition (cf. Kajii and Morris, 1998):

Definition 3.4.1. *Take any $\mu \in \mathcal{M}^{(n)}$, and consider a sequence $(\mu^k)_{k\in\mathbb{N}}$ in $\mathcal{M}^{(n)}$. The sequence $(\mu^k)_{k\in\mathbb{N}}$* converges strategically *to μ if for each profile v of local payoff functions and for each $\varepsilon > 0$ we have that*

$$\lim_{k\to\infty} \chi^*(\mu, \mu^k; v, \varepsilon) = 0.$$

In the next section, we give an example which illustrates the factors that are important for strategic convergence.

3.4.2 Example: Local investment

For reasons that will become clear shortly, let $\mathcal{N}$ be the set of integers that can be written as

$$N_\nu = (2^1 + 1)n_1 + (2^2 + 1)n_2 + \cdots + (2^\nu + 1)n_\nu$$

for some $\nu \in \mathbb{N}$, with $n_\nu = 1$ and for each $\ell \in \{1, \ldots, \nu - 1\}$, $n_\ell = 2^\ell\, n_{\ell+1}$.

Let $\nu \in \mathbb{N}$ and consider the following game. There is a set of $n = N_\nu$ players. Each player has two actions, S and R. Action S is the safe action. It always gives a payoff of 0, independent of a player's type or the actions and types of his neighbors. The payoffs to the risky action R depend on the actions of a player's neighbors in the network. More precisely, the payoffs to a player of type $t \in T, t > 0$, of action R when the action and type profiles of his neighbors are $a^{(t)} = (a_1^{(t)}, \ldots, a_t^{(t)}) \in A^t$ and $\theta^{(t)} = (\theta_1^{(t)}, \ldots, \theta_t^{(t)}) \in T^t$, respectively, are:

$$v_t\left(R, a^{(t)}, \theta^{(t)}\right) = \begin{cases} 3c & \text{if } a_\ell^{(t)} = R \text{ for all } \ell \in \{1, \ldots, t\}, \\ -c & \text{otherwise,} \end{cases}$$

where $c > 0$ is some constant. An interpretation of this game is that players need to decide whether to invest (play R) or not (play S). Investment is risky. Only if all his neighbors invest, a player gets a positive payoff from investing, otherwise he looses. Clearly, this is a game of (strict) strategic complements, since the incentives for a player to invest increase strictly when the number of neighbors who invest increases.

We consider two priors on $(\mathcal{G}^{(n)}, \mathcal{F}^{(n)})$, the *independent types prior* and the *core-periphery prior*. The core-periphery prior μ_{cp} assigns probability one to the isomorphism class of networks that, for $\ell \in \{1, \ldots, \nu\}$, consist of n_ℓ components with $2^\ell + 1$ players, of which one player—the core player—is connected to all other players, and the other 2^ℓ players—the peripheral players—are connected to the core player and to $2^{\ell-1} - 1$ peripheral players. Hence, the type (degree) of the core player in a component with $2^\ell + 1$ players is 2^ℓ, and the type of the peripheral players in such a component is $2^{\ell-1}$. We assume that each of the networks in the isomorphism class has equal probability. See Figure 3.2 for components that occur with positive probability when ν is at least 3. Note that we can only construct such networks when the number of players is an element of $\mathcal{N}$.

If we define $n_0 = n_{\nu+1} = 0$, it can easily be verified that the type distribution under μ_{cp} is given by $(\xi^{(n)}(t))_{t \in \mathbb{N}_0}$, where

$$\xi^{(n)}(t) := \begin{cases} \frac{1}{N_\nu}(n_{\log_2(t)} + 2t n_{\log_2(t)+1}) & \text{if } t \in \{1, 2, 4, \ldots, 2^\nu\}, \\ 0 & \text{otherwise.} \end{cases}$$

It can be readily checked that $\sum_{t \in \mathbb{N}_0} \xi^{(n)}(t) = 1$. In addition, it can be shown that for each $t \in \mathbb{N}_0$, $\xi^{(n)}(t)$ converges to some $\xi(t)$ when $\nu \to \infty$, with

$$\sum_{t \in \mathbb{N}_0} \xi(t) = 1, \qquad \sum_{t \in \mathbb{N}_0} t\xi(t) < \infty.$$

We refer to $(\xi(t))_{t \in \mathbb{N}_0}$ as the limiting (type) distribution. Furthermore, it is not hard to verify that for $\ell = 2, 3, \ldots, \nu - 1$ the conditional probability that a player of type

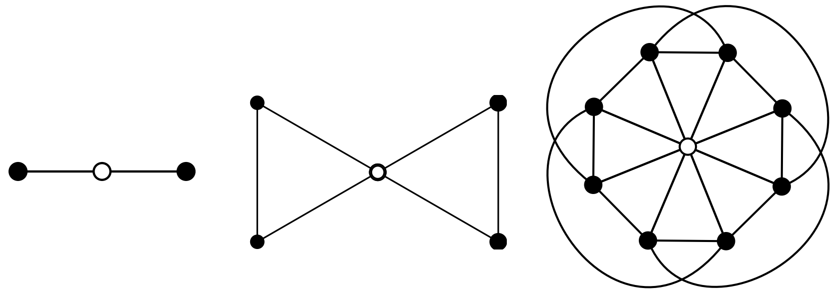


Figure 3.2. Components that occur with positive probability under the core-periphery prior when ν is at least 3. The core players are indicated with white dots, the peripheral players by black dots.

$t = 2^\ell$ is a core player is

$$\frac{1 \cdot n_\ell}{1 \cdot n_\ell + 2^{\ell+1} \cdot n_{\ell+1}} = \frac{1}{3},$$

independent of ℓ, where we have used that $n_\ell = 2^\ell n_{\ell+1}$. For future reference, note that a player with type 2^ℓ who is a core player interacts with players of type $t' = 2^{\ell-1}$. Similarly, the conditional probability that a player with type $t = 2^\ell$ is a peripheral player is

$$\frac{2^{\ell+1} \cdot n_{\ell+1}}{1 \cdot n_\ell + 2^{\ell+1} \cdot n_{\ell+1}} = \frac{2}{3}.$$

In that case, he interacts with players of type $t' = 2^{\ell+1}$.

We now define the independent types prior. We follow the literature (e.g. Galeotti et al., 2006; Jackson and Yariv, 2007; Sundararajan, 2005) by assuming that under the independent types prior, players believe that the type distribution is given by some fixed distribution and that players' types are independent. Here, we assume that the type distribution is $(\xi(t))_{t\in\mathbb{N}_0}$. Note that these assumptions require some bounded rationality on the part of players. for two reasons. Firstly, there exists no prior on the finite set $\mathcal{G}^{(n)}$ that gives rise to independent types. Moreover, under such beliefs, players assign positive probability to a player having a type (degree) that exceeds the number of players (minus one), which is clearly impossible. However, these assumptions can be justified in the following way. In Section 2.3.2, we have discussed a network belief system that gives rise to a prior on $\mathcal{G}^{(n)}$ that induces a degree distribution (type distribution) that is close to the limiting distribution $(\xi(t))_{t\in\mathbb{N}_0}$ such that degrees (types) are almost independent when the number of players is large. Furthermore, as shown in the next section, priors that are close in terms of the type distribution and the correlation among player types that they induce are similar in terms of game-theoretic predictions. This means that the results we would obtain under a type distribution close to $(\xi(t))_{t\in\mathbb{N}_0}$ and under almost independent types will be very similar in game-theoretic terms to the results we derive here for type distribution $(\xi(t))_{t\in\mathbb{N}_0}$ and independent types.

Hence, we assume that for each $t \in \mathbb{N}_0$, players' prior belief that the type of an arbitrary player is t is $\mu_{ind}(t) := \xi(t)$. We now derive the conditional probability that a fixed player has a given neighbor type profile, given his type. First note that if the probability that a player selected uniformly at random from the network has type (degree) t is $\xi(t)$, $t \in \mathbb{N}_0$, then the probability that a *neighbor* of a player selected uniformly at random from the network has degree t is proportional to $t\xi(t)$, as a player of degree t has t times more neighbors than a player of degree 1. Hence, for each $t \in \mathbb{N}_0$, the probability that the neighbor of a player selected uniformly at random from the network has type (degree) t

$$\frac{t\xi(t)}{\sum_{s\in\mathbb{N}_0} s\xi(s)}.$$

Then, for each neighbor type profile $\theta = (\theta_1, \dots, \theta_t) \in \Omega_K^t$, for all $k \in \mathbb{N}_0$, let $c_k(\theta)$ be the number of elements in θ that are equal to k. Define

$$M(\theta) := \frac{t!}{\prod_{k\in\mathbb{N}_0} c_k(\theta)!},$$

where we recall that $0! = 1$. That is, $M(\theta)$ is the multinomial coefficient corresponding to θ. Then, for each $t \in \mathbb{N}_0$ such that $\mu(t) > 0$, for each $\theta = (\theta_1, \dots, \theta_t) \in \Omega_K^t$, the conditional belief that a player's neighbor type profile is θ given that he has type t is

$$\mu_{ind}(\theta \mid t) = M(\theta) \prod_{\substack{s\in\mathbb{N}:\\ \xi(s)>0}} \left(\frac{s\xi(s)}{\sum_{s\in\mathbb{N}_0} r\xi(r)}\right)^{c_s(\theta)},$$

where we have used that $x^0 = 1$ for $x > 0$.[5] In words, the conditional distribution of neighbors' type, given that the "central" player has type t is given by the multinomial distribution. That is, the types of the player's neighbors are drawn in

[5] Note that $\mu_{ind}(t), t \in \mathbb{N}_0$, and $\mu_{ind}(\theta \mid t)$ for $t \in \mathbb{N}_0, \theta \in \Omega_K^t$, are *not* derived from some prior on a set of networks, as in the rest of the chapter.

t independent trials, with the probability that a neighbor has type s given by $\eta(s)$. It is important to note that $\mu_{ind}(\theta \mid t)$ does not depend on t.

Hence, the core-periphery prior and the independent types prior are very similar in terms of the type distribution they induce. Under the independent types prior, the type distribution is exactly $(\xi(t))_{t\in\mathbb{N}_0}$, while under the core-periphery prior it is close to $(\xi(t))_{t\in\mathbb{N}_0}$ (assuming that the number of players is large). However, the two priors are very different in the type correlation they induce. Under the independent types prior, types are independent. By contrast, under the core-periphery prior, players of type 2 only interact with players of type $1, 2$ and 4, players of type 4 only interact with players of type $2, 4$ and 8, and so on.

An important question is whether the two priors are similar from a game-theoretic perspective. It is easy to see that under both priors, there is a symmetric Bayesian-Nash equilibrium in which all players invest, regardless of their type, and a symmetric Bayesian-Nash equilibrium in which no player invests for any type he ends up having. There are also so-called threshold equilibria in which players invest if and only if their type is above or below some threshold. We show that there is a threshold equilibrium under the independent types prior such that there is no corresponding symmetric approximate equilibrium under the core-periphery prior and vice versa. Hence, the set of equilibria changes substantively when we change the correlation among player types.

We start by showing that there is a threshold equilibrium under the independent types prior such that there is no corresponding approximate equilibrium under the core-periphery prior. First, for $t, \bar{t} \in \mathbb{N}_0$, define

$$f(t;\bar{t}) := 3c \cdot \left(\frac{\sum_{t'\leq\bar{t}} t'\xi(t')}{\sum_{s\in\mathbb{N}_0} s\xi(s)}\right)^t - c \cdot \left(1 - \left(\frac{\sum_{t'\leq\bar{t}} t'\xi(t')}{\sum_{s\in\mathbb{N}_0} s\xi(s)}\right)^t\right).$$

When the number of players is sufficiently large, there is a unique $\bar{t} \in \{1, 2, \ldots, 2^{\nu-1}\}$ such that[6]

$$f(t;\bar{t}) \geq 0 \iff t \leq \bar{t}.$$

In that case, the expected payoffs under the independent types prior to a player of type t who chooses action R when other players follow the strategy of investing if and only if their type does not exceed the threshold $\bar{t}$ are given by $f(t;\bar{t})$. Then,

[6] Such a threshold exists. For each $\bar{t} \in \mathbb{N}_0$, $f(t;\bar{t})$ is declining in t, and for each $t \in \mathbb{N}_0$, $f(t;\bar{t}_1) > f(t;\bar{t}_2)$ whenever $\bar{t}_1 > \bar{t}_2$. Hence, there exists a unique $\bar{t} \in \mathbb{N}_0$ such that $f(t;\bar{t}) \geq 0$ if and only if $t \leq \bar{t}$; by choosing the number of players large enough, we have $\bar{t} \in \{1, 2, 3, \ldots, 2^{\nu-1}\}$.

it is easy to see that there is a symmetric Bayesian-Nash equilibrium under the independent types prior in which players invest if and only if their type is at most $\bar{t}$.[7]

By contrast, there does not exist a corresponding ε-equilibrium under the core-periphery prior for ε sufficiently small. To see this, suppose by contradiction that there would exists a threshold $\bar{t}$ such that players would invest if and only if their type is at most $\bar{t}$, and consider the lowest type $s_{min} := \min\{s \in \{1, 2, 4, \ldots, 2^{\nu}\} \mid s > \bar{t}\}$ that does not invest under this proposed equilibrium. The conditional probability that a player of type $t = 2^{\ell}$, where $\ell \in \{1, 2, \ldots, \nu - 1\}$, is a core player rather than a peripheral player, i.e., that all his neighbors invest under the proposed equilibrium, is

$$\frac{1 \cdot n_{\ell}}{1 \cdot n_{\ell} + 2^{\ell+1} \cdot n_{\ell+1}} = \frac{1}{3},$$

independent of t. Consequently, the interim expected payoffs to a player with type s_{min} of R under the proposed equilibrium are

$$\frac{3c}{3} - \frac{2c}{3} = \frac{c}{3} > 0.$$

Hence, for $\varepsilon < c/3$, it is an ε-best response to choose R for players with type s_{min}. But then, by the same argument, players with the next lowest type that does not invest under the proposed strategy will also find it profitable to invest (in terms of ε-best responses), and so on. Hence, there exists no ε-equilibrium under the core-periphery prior corresponding to the threshold equilibrium under the independent types prior if ε is sufficiently small.

We now show that there is a threshold equilibrium under the core-periphery prior which is not an (approximate) equilibrium under the independent types prior. Let $\hat{t} \in \{1, 2, 3, \ldots, 2^{\nu-1}\}$, and consider the symmetric strategy profile in which players invest if and only if their type is *at least* $\hat{t}$. As the interim expected payoffs of R to players of type t are declining in t for any such threshold strategy under the independent types prior, this strategy cannot be an ε-equilibrium under this prior for ε sufficiently small. However, such a strategy is a Bayesian-Nash equilibrium for any $\hat{t} \in \{1, 2, 3, \ldots, 2^{\nu-1}\}$ under the core-periphery prior. Fix $\hat{t}$, and suppose

[7] This result is not in contradiction with Proposition 2 of Galeotti et al. (2006), which shows that under independent types and strict strategic complements, every symmetric Bayesian-Nash equilibrium is monotone increasing in type (in the current setting, if low types invest, then high types invest, but not vice versa), as they assume that payoffs satisfy some additional property that is not satisfied by the current example.

players play R if and only if their type is at least $\hat{t}$. Consider a player of type $t = 2^\ell \geq \hat{t}$. With conditional probability

$$\frac{2^{\ell+1} \cdot n_{\ell+1}}{1 \cdot n_\ell + 2^{\ell+1} \cdot n_{\ell+1}} = \frac{2}{3},$$

all his neighbors play R, so that he earns a payoff of $3c$; with conditional probability

$$\frac{1 \cdot n_\ell}{1 \cdot n_\ell + 2^{\ell+1} \cdot n_{\ell+1}} = \frac{1}{3},$$

some neighbors play S, giving him a payoff of $-c$. His interim expected payoffs are thus $(6c/3) - (2c/3) > 0$, so that he cannot gain by deviating. Now consider a player of type $t < \hat{t}$. With probability 1, at least some of his neighbors play S, so his best response is to play S as well. Hence, under the core-periphery prior, there exists a threshold equilibrium in which players invest if and only if their type exceeds some threshold.

These examples show that strategy profiles that are Bayesian-Nash equilibria under one prior, may not be (approximate) equilibria under a prior which only differs from the first prior in the type correlation it induces. Note that if the number of players is sufficiently large, there are multiple threshold strategies that induce a symmetric Bayesian-Nash equilibrium under the core-periphery prior. By contrast, there is a unique threshold equilibrium strategy under the independent types prior. Hence, by choosing the constant c appropriately, we can find a threshold equilibrium under the core-periphery prior such that there is no symmetric approximate equilibrium under the independent types priors that is close to this threshold equilibrium in terms of ex ante expected payoffs. Hence, even though the priors are very close in terms of the type distribution they induce (for a large number of players), the strategic distance between them (given c and ε) can be large.

Similar examples can be constructed for other games, e.g. games with strategic substitutes (cf. Galeotti et al., 2006). In the next section, we show that the type distribution and the type correlation indeed determine the strategic distance between priors.

3.4.3 Main result

The example in the previous section suggests that differences in correlation among player types are an important determinant of the strategic distance between two priors. It is intuitive that also the type distribution induced by priors plays an important role. As we show in Lemma 3.4.3 below, closeness of priors in terms of the type distribution and the correlation among player types is equivalent to closeness in terms of the prior probabilities assigned to local events, i.e., events involving the type of a player and his neighbors. Hence, for $\mu, \mu' \in \mathcal{M}^{(n)}$, define

$$d^*(\mu, \mu') := \max_{F \in \mathscr{F}_K} |\mu(F) - \mu'(F)|.$$

That is, $d^*(\mu, \mu')$ measures the difference in probabilities assigned by μ and μ' to local events, or, equivalently (by Lemma 3.4.3), the difference in the type distribution and the type correlations induced by μ and μ'.

Theorem 3.4.2 establishes that convergence of priors in terms of prior probabilities assigned to local events is in fact necessary and sufficient for strategic convergence.

Theorem 3.4.2. *Let $\mu \in \mathcal{M}^{(n)}$ and let $(\mu^k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{M}^{(n)}$. Then, $(\mu^k)_{k \in \mathbb{N}}$ converges strategically to μ if and only if*

$$\lim_{k \to \infty} d^*(\mu, \mu^k) = 0.$$

The proof of Theorem 3.4.2 follows from Proposition 3.4.5 and Lemma 3.4.6. Proposition 3.4.5 shows that if two priors μ, μ' are close in terms of the prior probabilities assigned to local events, then for any Bayesian network game, for any symmetric Bayesian-Nash equilibrium of the game in which players hold the prior μ, there exists a symmetric approximate equilibrium in the game with prior μ' such that ex ante payoffs are similar. Proposition 3.4.5 uses Lemma 3.4.3 and Lemma 3.4.4.

Lemma 3.4.3. *Let $\mu \in \mathcal{M}^{(n)}$, and let $(\mu^k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{M}^{(n)}$. Let T' be the set of types $t \in T$ such that $\mu(t) > 0$ and $\mu^k(t) > 0$ for all $k \in \mathbb{N}$. Suppose that T' is nonempty, and that there exists $c > 0$ such that $\mu^k(t) \geq c$ for all $t \in T'$ uniformly over $k \in \mathbb{N}$. Then,*

$$\lim_{k \to \infty} \max_{F \in \mathscr{F}_K} |\mu(F) - \mu^k(F)| = 0 \iff \begin{cases} \lim_{k \to \infty} \max_{t \in T} |\mu(t) - \mu^k(t)| = 0, \\ \lim_{k \to \infty} \max_{t \in T', F \in \mathscr{F}_K} |\mu(F \mid t) - \mu^k(F \mid t)| = 0. \end{cases}$$

Proof. See Appendix 3.A. □

Lemma 3.4.4. *Let $\mu, \mu' \in \mathcal{M}^{(n)}$, and let $\gamma > 0$. Let v be a profile of local payoff functions with bound B. There exists $\delta > 0$ such that if σ is a symmetric Bayesian-Nash equilibrium of (μ, v) and $d^*(\mu, \mu') \leq \delta$, then there exists a symmetric $3\gamma B$-equilibrium σ' of the game (μ', v) with $\sigma'(\cdot \mid t) = \sigma(\cdot \mid t)$ for all $t \in T$ such that $\mu(t) > 0$ and $\mu'(t) > 0$.*

Proof. Define

$$S_{\mu,\mu'} := \{t \in T \mid \mu(t) > 0 \text{ and } \mu'(t) > 0\}$$

to be the set of types that occur with positive probability under both μ and μ'. Recall that $\hat{\sigma} = (\hat{\sigma}_t)_{t \in T}$ is defined by:

$$\forall t \in T, a \in A : \quad \hat{\sigma}_t(a) = \sigma_i(a \mid t) \quad \text{for any } i \in N.$$

Set $\hat{\sigma}'_t := \hat{\sigma}_t$ for all types $t \in S_{\mu,\mu'}$. For $t \notin S_{\mu,\mu'}$, take $\hat{\sigma}'_t$ such that $(\hat{\sigma}'_t)_{t \notin S_{\mu,\mu'}}$ induces a symmetric Bayesian-Nash equilibrium of the reduced game where each player $i \in N$ with a type $t \in S_{\mu,\mu'}$ is required to play $\hat{\sigma}'_t = \hat{\sigma}_t$. Such an equilibrium exists by Proposition 3.3.2. By construction, σ' is a best response for players with types $t \notin S_{\mu,\mu'}$. We need to show that σ' is a $3\gamma B$-best response for a type $t \in S_{\mu,\mu'}$. First, let

$$S_{\mu'} := \{t \in T \mid \mu'(t) > 0\}$$

be the set of types that have positive probability under μ'. Also, let $H \in \mathscr{F}_K$ be the event that a player interacts with at least one player with a type that has positive probability under μ' but not under μ, i.e.,

$$H = \bigcup_{t \in T} \left\{ (\theta_1, \dots, \theta_t) \in \Omega_K^t \;\middle|\; \exists \ell \in \{1, \dots, t\} : \theta_\ell \in S_{\mu'} \setminus S_{\mu,\mu'} \right\},$$

and let H^c be the complement (relative to Ω_K) of H. By definition,

$$\mu(H \mid t) = 0 \text{ for all } t \in S_{\mu,\mu'}. \tag{3.1}$$

By Lemma 3.4.3, there is a $\delta > 0$ such that if $d^*(\mu, \mu') \leq \delta$,

$$\max_{\substack{t \in S_{\mu,\mu'} \\ F \in \mathscr{F}_K}} |\mu(F \mid t) - \mu'(F \mid t)| \leq \gamma. \tag{3.2}$$

Combining (3.1) and (3.2) gives

$$\forall t \in S_{\mu,\mu'} : \quad \mu'(H \mid t) \leq \gamma. \tag{3.3}$$

Let $t \in S_{\mu,\mu'}$, and let $a, b \in A$ with $\hat{\sigma}'_t(a) > 0$. Then,

$$\begin{aligned} |\hat{\varphi}_t(a, \hat{\sigma}'; \mu') - \hat{\varphi}_t(b, \hat{\sigma}'; \mu')| \leq \sum_{\theta \in H} \mu'(\theta \mid t) \left| v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta) \right| + \\ \sum_{\theta \in H^c} \mu'(\theta \mid t) \left| v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta) \right|. \end{aligned} \tag{3.4}$$

By (3.3), recalling that the bound on v is B, the first sum in (3.4) is at most γB. To evaluate the second sum, first note that the neighbors of a player with neighbor type profile $\theta \in H^c$ play according to $\hat{\sigma}$. As a lies in the support of the symmetric Bayesian-Nash equilibrium σ of (μ, v),

$$\sum_{\theta \in \Omega_K} \mu(\theta \mid t)\, v_t(a, \hat{\sigma}_\theta, \theta) \geq \sum_{\theta \in \Omega_K} \mu(\theta \mid t)\, v_t(b, \hat{\sigma}_\theta, \theta). \tag{3.5}$$

Using that $\mu(\theta \mid t) = 0$ for all $\theta \in H$, we can rewrite (3.5) to find:

$$\begin{aligned}\sum_{\theta \in H^c} \mu(\theta \mid t)\,[v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta)] &\geq -\sum_{\theta \in H} \mu(\theta \mid t)\,[v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta)] \\ &= 0. \end{aligned} \tag{3.6}$$

Define $G_t := \{\theta \in H^c \mid \mu(\theta \mid t) - \mu'(\theta \mid t) > 0\}$ and let G_t^c be the complement of G_t relative to H^c. For notational simplicity, define

$$V_{\mu,\mu'}(a, b; \hat{\sigma}) := \left| \sum_{\theta \in H^c} \Big(\mu(\theta \mid t) - \mu'(\theta \mid t)\Big)\Big(v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta)\Big) \right|.$$

Using (3.2), it follows that

$$\begin{aligned} V_{\mu,\mu'}(a, b; \hat{\sigma}) &\leq \sum_{\theta \in G_t} \Big(\mu(\theta \mid t) - \mu'(\theta \mid t)\Big) \Big|v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta)\Big| + \\ &\qquad \sum_{\theta \in G_t^c} \Big(\mu'(\theta \mid t) - \mu(\theta \mid t)\Big) \Big|v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta)\Big| \\ &\leq 2\gamma B. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) gives

$$|\hat{\varphi}_t(a, \hat{\sigma}'; \mu') - \hat{\varphi}_t(b, \hat{\sigma}'; \mu')| \leq 3\gamma B. \qquad \square$$

Proposition 3.4.5. *Let $\mu, \mu' \in \mathcal{M}^{(n)}$, and fix $\gamma > 0$. Let v be a profile of local payoff functions with bound B, and let $\delta > 0$ as in Lemma 3.4.4. Let $\eta \in (0, \delta]$, and suppose that*

$$d^*(\mu, \mu') \leq \eta.$$

Then, if σ is a symmetric Bayesian-Nash equilibrium of the game (μ, v), there exists a symmetric $3\gamma B$-equilibrium σ' of the game (μ', v) such that

$$|\hat{\Phi}(\hat{\sigma}; \mu) - \hat{\Phi}(\hat{\sigma}'; \mu')| \leq 4\eta B,$$

where $\hat{\sigma} = (\hat{\sigma}_t)_{t \in T}$ and $\hat{\sigma}' = (\hat{\sigma}'_t)_{t \in T}$ are defined by $\hat{\sigma}_t = \sigma_i(\cdot \mid t)$ and $\hat{\sigma}'_t = \sigma'_i(\cdot \mid t)$ for any $i \in N$ for all $t \in T$.

Proof. Let σ be a symmetric Bayesian-Nash equilibrium of (μ, v). By Lemma 3.4.4, there exists a symmetric $3\gamma B$-equilibrium σ' of the game (μ', v) such that $\hat{\sigma}'_t = \hat{\sigma}_t$ for $t \in T$ such that $\mu(t) > 0$ and $\mu'(t) > 0$. Define

$$G := \{\theta \in \Omega_K \mid \mu(\theta) - \mu'(\theta) > 0\},$$

and let G^c be the complement of G relative to Ω_K. Define the function $\zeta : \Omega_K \to T$ by $\zeta(\theta) = t$ whenever $\theta \in \Omega_K^t$. That is, the function ζ gives the type of a player for each possible neighbor type profile he may have. Then,

$$\begin{aligned} |\hat{\Phi}(\hat{\sigma}; \mu') - \hat{\Phi}(\hat{\sigma}; \mu)| &\leq \sum_{\theta \in G} (\mu(\theta) - \mu'(\theta)) \sum_{a \in A} \hat{\sigma}_{\zeta(\theta)}(a) \, |v_{\zeta(\theta)}(a, \hat{\sigma}_\theta, \theta)| \, + \\ &\qquad \sum_{\theta \in G^c} (\mu'(\theta) - \mu(\theta)) \sum_{a \in A} \hat{\sigma}_{\zeta(\theta)}(a) \, |v_{\zeta(\theta)}(a, \hat{\sigma}_\theta, \theta)| \\ &\leq 2\eta B. \end{aligned} \tag{3.8}$$

Also, define

$$\begin{aligned} F_{\mu'} &:= \{\theta \in \Omega_K \mid \mu'(\zeta(\theta)) > 0\}, \\ F_{\mu,\mu'} &:= \{\theta \in \Omega_K \mid \mu(\zeta(\theta)) > 0 \text{ and } \mu'(\zeta(\theta)) > 0\}. \end{aligned}$$

Then, as $\mu(F_{\mu'} \setminus F_{\mu,\mu'}) = 0$ by definition,

$$\mu'(F_{\mu'} \setminus F_{\mu,\mu'}) \leq \eta.$$

Recalling that $\hat{\sigma}'_t = \hat{\sigma}_t$ for t such that $\mu(t) > 0$ and $\mu'(t) > 0$, this yields

$$\begin{aligned} &|\hat{\Phi}(\hat{\sigma}'; \mu') - \hat{\Phi}(\hat{\sigma}; \mu')| \\ &\leq \sum_{\theta \in F_{\mu,\mu'}} \mu'(\theta) \left| \sum_{a \in A} \hat{\sigma}'_{\zeta(\theta)}(a) v_{\zeta(\theta)}(a, \hat{\sigma}'_\theta, \theta) - \sum_{a \in A} \hat{\sigma}_{\zeta(\theta)}(a) v_{\zeta(\theta)}(a, \hat{\sigma}_\theta, \theta) \right| + \\ &\qquad \sum_{\theta \in F_{\mu'} \setminus F_{\mu,\mu'}} \mu'(\theta) \left| \sum_{a \in A} \hat{\sigma}'_{\zeta(\theta)}(a) v_{\zeta(\theta)}(a, \hat{\sigma}'_\theta, \theta) - \sum_{a \in A} \hat{\sigma}_{\zeta(\theta)}(a) v_{\zeta(\theta)}(a, \hat{\sigma}_\theta, \theta) \right| \\ &= \sum_{\theta \in F_{\mu'} \setminus F_{\mu,\mu'}} \mu'(\theta) \left| \sum_{a \in A} \hat{\sigma}'_{\zeta(\theta)}(a) v_{\zeta(\theta)}(a, \hat{\sigma}'_\theta, \theta) - \sum_{a \in A} \hat{\sigma}_{\zeta(\theta)}(a) v_{\zeta(\theta)}(a, \hat{\sigma}_\theta, \theta) \right| \\ &\leq 2\eta B. \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9) gives

$$|\hat{\Phi}(\hat{\sigma}; \mu) - \hat{\Phi}(\hat{\sigma}'; \mu')| \leq 4\eta B. \qquad \square$$

Proposition 3.4.5 establishes the sufficiency of our condition for strategic convergence. Lemma 3.4.6 below shows that the condition that d^* should be small is also necessary for strategic convergence.

Lemma 3.4.6. *Let $\delta \in [0, 1]$, and let $\mu, \mu' \in \mathcal{M}^{(n)}$. If*

$$d^*(\mu, \mu') > \delta,$$

then there exists a profile v of local payoff functions with bound $B = 1$ and a symmetric Bayesian-Nash equilibrium σ of the game (μ, v) such that for any symmetric δ-equilibrium σ' of (μ', v), it holds that

$$|\hat{\Phi}(\hat{\sigma}; \mu) - \hat{\Phi}(\hat{\sigma}'; \mu')| > \delta,$$

where $\hat{\sigma} = (\hat{\sigma}_t)_{t \in T}$ and $\hat{\sigma}' = (\hat{\sigma}'_t)_{t \in T}$ are defined by $\hat{\sigma}_t = \sigma_i(\cdot \mid t)$ and $\hat{\sigma}'_t = \sigma'_i(\cdot \mid t)$ for any $i \in N$ for all $t \in T$.

Proof. By assumption, there exists a set of neighbor type profiles $F \in \mathscr{F}_K$ such that $|\mu(F) - \mu'(F)| > \delta$. For each $t \in T, t > 0, a \in A, a^{(t)} \in A^t$ and $\theta \in \Omega_K^t$, let

$$v_t(a, a^{(t)}, \theta) = \begin{cases} 1 & \text{if } \theta \in F, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$|\hat{\Phi}(\hat{\sigma}; \mu) - \hat{\Phi}(\hat{\sigma}'; \mu')| > \delta$$

for *any* two symmetric strategy profiles $\sigma, \sigma' \in \Sigma^n$. □

We can now prove Theorem 3.4.2:

Proof. (If) Let v be a profile of local payoff functions with bound B. Let $\gamma > 0$ be arbitrarily small, and let $\delta > 0$ be as in Lemma 3.4.4. Take any $\varepsilon \in (0, \delta]$. Since $d^*(\mu, \mu^k) \to 0$ as $k \to \infty$, it holds that $d^*(\mu, \mu^k) \leq \varepsilon$ for all k sufficiently large. Hence, by Proposition 3.4.5,

$$\chi^*(\mu, \mu^k; v, 3B\gamma) \leq 4B\varepsilon$$

for k sufficiently large. That is, if $d^*(\mu, \mu^k) \to 0$ as $n \to \infty$, then, for any v and any $c > 0$, we have $\chi^*(\mu, \mu^k; v, c) \to 0$ as $k \to \infty$.

(Only if) Let $\mu, \mu' \in \mathcal{M}^{(n)}$. For $\delta \in [0, 1)$, if $d^*(\mu, \mu') > \delta$, then, by Lemma 3.4.6, there exists a profile of local payoff functions v with bound $B = 1$ and a symmetric Bayesian-Nash equilibrium $\sigma \in \Sigma^n$ of (μ, v) such that for any symmetric δ-equilibrium $\sigma' \in \Sigma^n$ of (μ', v), $|\hat{\Phi}(\hat{\sigma}; \mu) - \hat{\Phi}(\hat{\sigma}'; \mu')| > \delta$. □

Theorem 3.4.2 shows that for two priors to be close in a strategic sense, it is necessary and sufficient for them to be close in terms of prior probabilities they assign to local events, i.e., events that involve the types of a player and his neighbors. This result has two important implications. Firstly, this result means that in order to explore the full range of strategic outcomes in Bayesian network games, it is sufficient to vary the type distribution and the correlation among player types. Hence, on the one hand, it suggests that varying the type distribution, as has been the focus of much of the literature so far, is often not enough. On the other hand, it limits the set of priors that one needs to consider. We show that priors need only be varied along two dimensions. A second important implication is that we can interpret a Bayesian network game as a set of overlapping "local games", and that we do not need to concern ourselves with the nonlocal features of network belief systems. Refer back to the networks in Figure 3.1(a) and (b), and consider two priors, one that assigns positive probability only to networks that are isomorphic to the network in Figure 3.1(a), with each of the networks in this isomorphism class having equal probability, and the other assigning positive probability only to networks isomorphic to the network in Figure 3.1(b), with each of the networks in this isomorphism class having equal probability. Theorem 3.4.2 tells us that these priors are identical in terms of their game-theoretic predictions, even though they are very different in terms of the networks they predict.

Hence, by exploiting the symmetry of the game and the local features of the payoff functions, it is possible to weaken the conditions of Kajii and Morris (1998) for this particular class of Bayesian games. Kajii and Morris (1998) show that for general Bayesian games with finite type sets, priors need to be close in terms of the prior probabilities they assign to all possible events. By contrast, we only require that priors are close in terms of the prior probabilities assigned to *local* events. While it is not surprising that we can weaken the general result of Kajii and Morris (1998) for a subclass of games, it yields the useful insight that we can treat Bayesian network games as a collection of overlapping local games, and that the important features of priors in terms of strategic outcomes are the type distribution and correlation among player types that they induce.

We end this section with a discussion of our framework and our assumptions. Firstly, in the current chapter, we have focused on symmetric equilibria, as this is the focus of much of the literature on Bayesian network games (e.g. Galeotti et al., 2006; Jackson and Yariv, 2007; Sundararajan, 2005). It is possible to derive similar results for general Bayesian-Nash equilibria, though it will not be possible to exploit the symmetry of the game as we have done here. If one would consider general equilibria, results similar to those of Kajii and Morris (1998) would be obtained.

Secondly, our definition of strategic closeness requires that ex ante expected payoffs be close in equilibria under two priors, i.e., we focus on payoff continuity. An alternative continuity notion would require that with high probability, a player and his neighbors follow the same strategies under the two priors (cf. Monderer and Samet, 1996). Indeed, from the proof of Proposition 3.4.5, it follows that if two priors are close in terms of the measure d^*, then for each symmetric Bayesian-Nash equilibrium under one of the priors, there exists an approximate equilibrium under the other prior which coincides with the first equilibrium for all types that have positive probability under both priors, i.e., there is also continuity in terms of strategies. However, in the current setting (unlike in the setting of e.g. Monderer and Samet, 1996), we also have to consider the difference in prior probabilities that players have a given type under the two priors in order to ensure that the two priors give rise to similar outcomes from a player's ex ante perspective. Hence, the appropriate definition of strategic closeness in the current setting considers differences in ex ante expected payoffs.

Thirdly, while we study the general case in which payoffs depend on the actions and types of a player and his neighbors, one could also consider the special case in which a player's payoffs depend only on his own action and type and on the actions of his neighbors, and not on his neighbors' types. Obviously, for this subclass of games, the condition we derived for strategic convergence is still sufficient, though it may not be necessary. Our conjecture is that the condition cannot be weakened substantially for this subclass of games.

Finally, in line with the literature, we have studied games in which a player's payoffs only depend on the actions and types of his direct neighbors. Our result can easily be generalized to the case where a player's payoff depends on the actions and types of those within k steps in the network, for some $k \in \mathbb{N}$. Of course, when k increases, the condition for two priors to be close becomes more strict, ultimately recovering the condition of Kajii and Morris (1998) that priors need to be close in terms of the prior probabilities they assign to global events. Indeed, when the payoffs to a player depend on the actions and types of all others in the network (even on the actions and types of those with whom he is not directly connected), the game can be alternatively modeled as a standard Bayesian game, with some suitable restrictions on payoffs.

3.5 Conclusions

Networks are ubiquitous in economics, and they can have a large effect on economic outcomes. The current chapter considers a setting in which players are located on a network and play a fixed game with their neighbors. Players have incomplete information on the network structure. They have a common prior on a given class of networks, and, in addition, they have some local information on the network structure. Given the complexity of many social and economic networks, it is important to study whether game-theoretic predictions are sensitive to assumptions on players' beliefs.

In the current chapter, we have studied the conditions that are necessary and sufficient for two (common) priors to be close in a strategic sense. More specifically, we have studied the conditions under which for any Bayesian network game in which players hold one of these priors, for any symmetric Bayesian-Nash equilibrium in that game, there is a symmetric approximate equilibrium in the associated game with the other prior such that ex ante expected payoffs are close under the two equilibria. Our main result (Theorem 3.4.2) states a necessary and sufficient condition for two priors to be close in this sense is that they be close in terms of the prior probabilities they assign to local events, i.e., events involving the type of an arbitrary player and his neighbors. An equivalent condition is that two priors be close in terms of the type distribution and the correlation among player types they induce (Lemma 3.4.3). Hence, the essential features of a prior in Bayesian network games are the type distribution and the type correlation it induces.

This result suggests that one needs to go beyond priors with independent types, which has been the focus of much of the literature so far. We have illustrated this point in Section 3.4.2, where we show that priors with the same type distribution can give rise to very different equilibria in a simple game, depending on the correlation among player types. The current result also puts restrictions on the set of priors one needs to consider. We show that one only needs to vary the type distribution and the correlation among types.

There are several directions for further research. Firstly, the current result indicates that it is important to systematically assess the effect of varying the type distribution and the correlations among player types on game-theoretic outcomes. While several authors study the effect of varying the type distribution in specific games (e.g. Jackson and Yariv, 2007; Sundararajan, 2005), there is little work on the effect on game-theoretic outcomes of changing the correlation among players' types. Galeotti et al. (2006) analyze the effect of some specific changes in the type distribution and the correlation among player types in certain classes of games. However, there is no systematic exploration of the effect of changing the type correlations. Such an analysis will not be easy. There are two prime difficulties. The first is that it is not clear how type correlation should be measured. Galeotti et al. (2006) define the concepts of positive and negative association, but these only seem to capture some dimensions of type correlation. The second is that it is hard to define suitable random network models in which the appropriate dimensions of type correlations can be varied continuously. Of the different candidate classes of random network models, especially those featuring community structures seem promising. Social and economic networks are typically structured in communities (e.g. Copič et al., 2005; Palla et al., 2005). The community structure induces a nonzero correlation among players' degrees (see Newman and Park, 2003, for a discussion). Random network models with a community structure, such as the one discussed in Chapter 5 thus seem to be natural candidates.

A second natural extension of the current work would be to allow for uncertainty over the network size. The observation of Myerson (1998) that in some contexts, it is reasonable to assume that players are uncertain about the number of other players in the game holds *a fortiori* for network games, as in these games, players only interact with a small subset of players and have no direct information about the players they do not interact with. In the next chapter, we therefore allow for uncertainty over the network size, and show that this gives rise to qualitatively different results, as in that case, players' higher order beliefs play an important role.

3.A Proofs

3.A.1 Proof of Proposition 3.3.2

Define the strategic game

$$G := \langle N, (M_i)_{i\in N}, (\check{\Phi}_i(\cdot;\mu))_{i\in N}\rangle,$$

where for each $i \in N$, the set of pure strategies M_i is the set of maps $m_i : T \to A$. Hence, we have $M_i =: M$ for all $i \in N$, and the set M is finite. For each $i \in N$, the payoff function $\check{\Phi}_i(\cdot;\mu)$ is defined by:

$$\forall m \in M^n : \quad \check{\Phi}_i(m;\mu) := \sum_{g\in\mathcal{G}^{(n)}} \mu(g)\, v_{\tau_i(g)}(m_i(\tau_i(g)), (m_j(\tau_j(g)))_{j\in N_i(g)}, (\tau_j(g))_{j\in N_i(g)}).$$

Mixed strategies are obtained by randomizing over strategies in the set M. Denote the set of mixed strategies in G by $\Delta(M)$. Payoffs can be extended to mixed strategies in the standard way. That is, for each $i \in N$, $\beta \in (\Delta(M))^n$,

$$\check{\Phi}_i(\beta;\mu) := \sum_{g\in\mathcal{G}^{(n)}} \mu(g) \sum_{m_i\in M} \beta_i(m_i) \sum_{m\in M^{\tau_i(g)}} \left(\prod_{j\in N_i(g)} \beta_j(m_j)\right) v_{\tau_i(g)}(m_i(\tau_i(g)), m_{N_i(g)}, \tau_{N_i(g)}),$$

where we have defined $m_{N_i(g)} := (m_j(\tau_j(g)))_{j\in N_i(g)}$ and $\tau_{N_i(g)} := (\tau_j(g))_{j\in N_i(g)}$.

The proof now follows from two steps:

Step 1: There exists $\beta = (\beta_j)_{j\in N} \in (\Delta(M))^n$ with $\beta_i = \beta_j$ for all $i, j \in N$ such that for all $i \in N$,

$$\check{\Phi}_i(\beta;\mu) \geq \check{\Phi}_i(\beta_i', \beta_{-i};\mu)$$

for all $\beta_i' \in \Delta(M)$.

Proof of Step 1: The set $\Delta(M)$ is a nonempty, convex and compact subset of the Euclidean space $\mathbb{R}^{|M|}$, and, by standard arguments, $\check{\Phi}_i(\cdot;\mu)$ is continuous in $\beta = (\beta_i, \beta_{-i})$ and quasiconcave in β_i. Furthermore, the game G is symmetric by Assumption 3.A and the symmetry of the payoff functions. Define the correspondence $\mathcal{B}$ on $\Delta(M)$ by:

$$\forall \beta \in \Delta(M): \quad \mathcal{B}(\beta) := \arg\max_{\alpha\in\Delta(M)} \check{\Phi}_i(\alpha, \beta, \ldots, \beta;\mu) \quad \text{for any } i \in N,$$

i.e., $\mathcal{B}(\beta)$ is the set of best responses (mixed strategies) of a player when other players play according to β. By standard arguments, the correspondence $\mathcal{B}$ is nonempty, convex-valued, and upper-hemicontinuous (e.g. Fudenberg and Tirole, 1991, pp. 29–30). Hence, by Kakutani's fixed point theorem (e.g. Ok, 2007, p. 331), a fixed point exists for $\mathcal{B}$, i.e., there exists $\beta \in \Delta(M)$ such that $\beta \in \mathcal{B}(\beta)$.

Step 2: Let $\beta = (\beta_j)_{j\in N} \in (\Delta(M))^n$ with $\beta_i = \beta_j$ for all $i, j \in N$ be such that for all $i \in N$,

$$\check{\Phi}_i(\beta;\mu) \geq \check{\Phi}_i(\beta_i', \beta_{-i};\mu)$$

for all $\beta_i' \in \Delta(M)$, and define $\sigma = (\sigma_j)_{j\in N} \in \Sigma^n$ by:

$$\forall i \in N, a_i \in A, t_i \in T: \quad \sigma_i(a_i \mid t_i) = \sum_{\substack{m_i\in M:\\ m_i(t_i)=a_i}} \beta_i(m_i).$$

Then, for each $i \in N$,

$$\Phi_i(\sigma; \mu) \geq \Phi_i(\sigma'_i, \sigma_{-i}; \mu)$$

for all $\sigma'_i \in \Sigma$.

Proof of Step 2: From substituting the relevant expressions, we obtain

$$\forall i \in N : \Phi_i(\sigma; \mu) = \check{\Phi}_i(\beta; \mu).$$

Suppose by contradiction that there exists $i \in N$ such that for some $\sigma'_i \in \Sigma$,

$$\Phi_i(\sigma'_i, \sigma_{-i}; \mu) > \Phi_i(\sigma; \mu).$$

Define $\beta'_i \in \Delta(M)$ by:

$$\forall m_i \in M : \quad \beta'_i(m_i) := \prod_{t_i \in T} \sigma'_i(m_i(t_i) \mid t_i).$$

Then, again by substitution,

$$\check{\Phi}(\beta'_i, \beta_{-i}; \mu) = \Phi_i(\sigma'_i, \sigma_{-i}; \mu) > \Phi_i(\sigma; \mu) = \check{\Phi}(\beta; \mu),$$

which contradicts that no player in G can gain by deviating unilaterally from β.

From Step 1 and 2 it follows that there exists $\sigma = (\sigma_j)_{j \in N} \in \Sigma^n$ such that for all $i \in N$,

$$\Phi_i(\sigma; \mu) \geq \Phi_i(\sigma'_i, \sigma_{-i}; \mu)$$

for all $\sigma'_i \in \Sigma$, and $\sigma_i = \sigma_j$ for all $i, j \in N$, i.e., there exists a symmetric Bayesian-Nash equilibrium for (μ, v). □

3.A.2 Proof of Lemma 3.4.3

(If) Suppose that $\lim_{k \to \infty} \max_{F \in \mathscr{F}_K} |\mu(F) - \mu^k(F)| = 0$. Then, clearly, for all $k \in \mathbb{N}$,

$$\max_{t \in T} |\mu(t) - \mu^k(t)| \leq \max_{F \in \mathscr{F}_K} |\mu(F) - \mu^k(F)|,$$

and hence

$$\lim_{k \to \infty} \max_{t \in T} |\mu(t) - \mu^k(t)| = 0.$$

Also, for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\max_{\substack{t\in T'\\ F\in\mathscr{F}_K}}|\mu(F \mid t) - \mu^k(F \mid t)| \\
&\qquad = \max_{\substack{t\in T'\\ F\in\mathscr{F}_K}}\left|\frac{\mu(F\cup\Omega_K^t)}{\mu(\Omega_K^t)} - \frac{\mu^k(F\cup\Omega_K^t)}{\mu^k(\Omega_K^t)}\right| \\
&\qquad = \max_{\substack{t\in T'\\ F\in\mathscr{F}_K}}\left|\frac{\mu(F\cup\Omega_K^t)}{\mu^k(\Omega_K^t)} - \frac{\mu^k(F\cup\Omega_K^t)}{\mu^k(\Omega_K^t)} + \frac{\mu(F\cup\Omega_K^t)}{\mu(\Omega_K^t)} - \frac{\mu(F\cup\Omega_K^t)}{\mu^k(\Omega_K^t)}\right| \\
&\qquad \le \max_{\substack{t\in T'\\ F\in\mathscr{F}_K}}\frac{1}{\mu^k(t)}\left|\mu(F\cup\Omega_K^t) - \mu^k(F\cup\Omega_K^t)\right| + \max_{\substack{t\in T'\\ F\in\mathscr{F}_K}}\frac{\mu(F \mid t)}{\mu^k(t)}\left|\mu(t) - \mu^k(t)\right| \\
&\qquad \le \left(\frac{2}{c}\right)\cdot\max_{F\in\mathscr{F}_K}|\mu(F) - \mu^k(F)|
\end{aligned}$$

where we have used the triangle inequality for the first inequality. Hence,

$$\lim_{k\to\infty}\max_{\substack{t\in T'\\ F\in\mathscr{F}_K}}|\mu(F \mid t) - \mu^k(F \mid t)| = 0.$$

(Only if) Suppose that

$$\lim_{k\to\infty}\max_{t\in T'}|\mu(t) - \mu^k(t)| = 0 \tag{3.10}$$

and

$$\lim_{k\to\infty}\max_{\substack{t\in T'\\ F\in\mathscr{F}_K}}|\mu(F \mid t) - \mu^k(F \mid t)| = 0. \tag{3.11}$$

Fix $\varepsilon > 0$. For each $F \in \mathscr{F}_K$ and each $k \in \mathbb{N}$, it holds that

$$\begin{aligned}
|\mu(F) - \mu^k(F)| &= \left|\sum_{t\in T'}\left[\mu(F \mid t) - \mu^k(F \mid t)\right]\mu(t) + \sum_{t\in T'}\mu^k(F \mid t)[\mu(t) - \mu^k(t)]\right| \\
&\le \sum_{t\in T'}|\mu(F \mid t) - \mu^k(F \mid t)|\mu(t) + \sum_{t\in T'}\mu^k(F \mid t)|\mu(t) - \mu^k(t)|.
\end{aligned}$$

By (3.10) and (3.11), there exists $Q \in \mathbb{N}$ such that for all $t \in T'$, $k > Q$ implies that

$$|\mu(F \mid t) - \mu^k(F \mid t)| < \varepsilon\cdot\left(\frac{c}{1+c}\right)$$

and

$$|\mu(t) - \mu^k(t)| < \varepsilon\cdot\left(\frac{c}{1+c}\right).$$

Hence, for $k > Q$,

$$\begin{aligned}
|\mu(F) - \mu^k(F)| &\leq \varepsilon \cdot \left(\frac{c}{1+c}\right)\left[\sum_{t\in T'} \mu(t) + \sum_{t\in T'} \mu^k(F \mid t)\right] \\
&\leq \varepsilon \cdot \left(\frac{c}{1+c}\right) \cdot \left[1 + \frac{1}{c}\sum_{t\in T'} \mu^k(F \mid t)\mu^k(t)\right] \\
&\leq \varepsilon \cdot \left(\frac{c}{1+c}\right) \cdot \left[1 + \frac{1}{c}\right] \\
&= \varepsilon,
\end{aligned}$$

and hence $\lim_{k\to\infty}|\mu(F) - \mu^k(F)| = 0$. As this holds for all $F \in \mathscr{F}_K$, we have

$$\lim_{k\to\infty} \max_{F\in\mathscr{F}_K} |\mu(F) - \mu^k(F)| = 0. \qquad \square$$

4 Higher order beliefs in network games

Summary

In this chapter, which is based on Kets (2007a), we consider network games in which players have incomplete information on the network structure, as in the previous chapter. In contrast with Chapter 3, we allow for uncertainty over the network size. As in the previous chapter, we study the sensitivity of game-theoretic predictions to the specification of beliefs. We show that two priors are close in a strategic sense if and only if (*i*) they assign similar probabilities to all local events; (*ii*) with high probability, a player believes, given his type, that his neighbors' conditional beliefs are close under the two priors, and that his neighbors believe, given their type, that. . . the conditional beliefs of their neighbors are close, and so on, for any number of iterations. The reason that we obtain different conditions than in Chapter 3 is that priors may now be sensitive to small probability events through players' conditional beliefs.

4.1 Introduction

The last few decades, a wealth of empirical studies has emerged that has documented how social and economic networks shape behavior and determine economic outcomes. Since the seminal work of Coleman et al. (1966) on the diffusion of technologies and of Granovetter (1974) on job contact networks, it is widely recognized that networks—be they networks of firms, countries, or individuals—act as conduits for information, knowledge, and capital, and that they shape individuals' behavior (e.g. Conley and Udry, 2005; Fafchamps and Lund, 2003; Glaeser et al., 1996; Powell et al., 1996; Tucker, 2005).

Social and economic networks are often large and complex, and evolve rapidly over time, so that it is natural to assume that agents belonging to the network will not know its precise structure. In a setting where agents on a network interact strategically with their neighbors, we therefore have to model the beliefs they have over the network structure. Considering a player's beliefs over his direct neighborhood may not be sufficient: a player's optimal action depends on the actions he expects his neighbors to take, but what actions they take will depend on the actions they expect their neighbors to take, and so on. This means that a player needs not only have beliefs over his neighbors, but also on the beliefs of his neighbors, and on the beliefs of his neighbors on the beliefs of their neighbors, etcetera.

In the current chapter, we consider a setting in which agents are located on a network and interact strategically with their neighbors under incomplete information on the network structure. In particular, there may be uncertainty about the size of the network. More specifically, players have a common prior over the network structure, and, in addition, they have some local information: they are informed of the number of neighbors they have in the network, i.e., their type is their *degree*. Interactions are local: a player's payoff only depends on his own action and type, and the actions and types of his direct neighbors in the network. This defines the class of *network games of incomplete information*. The difference with the class of Bayesian network games studied in the previous chapter is that in network games of incomplete information, there may be uncertainty over the network size. In the context of network games, this is a natural assumption, as players only interact with a small subset of players. Because the size of the network is not known, the set of players is not common knowledge, so that network games of incomplete information are not Bayesian games.[1]

As in the previous chapter, we study the sensitivity of game-theoretic predictions to the specification of players' beliefs on the network. We ask under what conditions on two priors it is the case that for any network game, for any equilibrium in the game with one prior, there is an approximate equilibrium in the game with the other prior, such that ex ante expected payoffs are close under the two equilibria. If that is the case, we say that the two priors are *close in a strategic sense*. We study the necessary and sufficient conditions for two priors to be close in a strategic sense, that is, we study a type of lower hemicontinuity of the correspondence of (interim) approximate equilibria in network games (see Engl, 1995, for a discussion of different continuity concepts).

The difference with the setting studied in the previous chapter, where we studied a similar question, is that we now allow for uncertainty over the network size. This seemingly innocuous extension turns out to have a large impact: if we allow for uncertainty over the network size, player's higher order beliefs

[1] In games with uncertainty over the player set in which all players interact directly, it is possible to obtain the Bayesian setting as a special case of the case with population uncertainty. When the number of players is fixed with probability one, it is possible to link a player's type with his identity, thus recovering the standard Bayesian setting (Milchtaich, 2004; Myerson, 1998). This is not possible in the current setting, where the interpretation of a player's type is given: a player's type is his degree. However, one could define a class of network games with two-dimensional types in which one dimension is a player's degree and the second is a dummy that can be interpreted as a player's identity. Then both the class of network games with incomplete information and the class of Bayesian network games are equivalent to special cases of this more general framework.

play an important role in the sense that priors may now be *sensitive to small probability events*: events that have small probability ex ante can have a large effect on outcomes through players' conditional beliefs. Suppose that there is a set of types for which conditional beliefs are very different under two priors, leading players with these types to choose different strategies under the two priors, and suppose that this set has small probability ex ante. Then, this set of types may "infect" the behavior of other types through players' conditional beliefs: a player may think it is likely, given his type, that his neighbors think it is likely, given their type,. . . the conditional beliefs of their neighbors are very different. Even if the set of types for which conditional beliefs are very different under the two priors has small probability ex ante, the set of infected types may have large prior probability, so that equilibria (in terms ex ante expected payoffs) will be very different under the two priors.

This is reflected in our main result (Theorem 4.5.3), which states that two priors are close in a strategic sense if and only if (*i*) the priors assign similar probabilities to all local events, i.e., events that involve a player and his neighbors, and (*ii*) with high probability, a player believes, given his type, that his neighbors' conditional beliefs are close under the two priors, and that his neighbors believe, given their type, that. . . the conditional beliefs of their neighbors are close, and so on, for any number of iterations. The first condition is analogous to the condition we derived in the previous chapter when we studied strategic convergence in Bayesian network games. The second condition is new, and rules out the situation described above, where a set of players with small ex ante probability affects the behavior of a set of types with large prior probability.

The reason that we require condition (*ii*) for strategic closeness in network games of incomplete information but not in Bayesian network games lies in the uncertainty over the network size. In Section 4.5.2 we show that a necessary condition for a prior to be sensitive to small probability events is that the set of types which has positive probability is countably infinite, and that types are not independent. When the set of types with positive probability is finite or when types are independent, closeness of the two priors in terms of the prior probabilities assigned to local events (condition (*i*)) implies that there is a sufficiently large set of players whose conditional beliefs are close (condition (*ii*)). Hence, priors in Bayesian network games are not sensitive to small probability events, while priors in network games of incomplete information can be.

Theorem 4.5.3 has an important result, for two reasons. Firstly, it underlines the need for careful modeling, as seemingly innocuous assumptions about players' beliefs can have large ramifications. Secondly, and perhaps more importantly, Theorem 4.5.3 provides insights in the way higher order beliefs affect behavior in network games. Interestingly, condition (*ii*) can also be stated in terms of correlation among types: an equivalent formulation of condition (*ii*) is that the set of types for which conditional beliefs are close under two priors must have high probability and is sufficiently cohesive in the sense that with high conditional probability, a type in that set interacts only with types in that set that, with high conditional probability, only interact with types in that set, and so on.

Compare this formulation of our result to the results of Morris (2000) on contagion. Morris (2000) studies interactions on a fixed network, i.e., in his setting, there is complete information on the network structure. He finds that behavior does *not* spread contagiously on such a network starting from a finite set of players X by myopic best-reply dynamics if and only if the network of players not belonging to X contains a large group of players Y that is sufficiently cohesive, in the sense that players from Y interact mostly with other players from Y, who in turn interact primarily with other players from Y, and so on.

Our result is a direct stochastic analogue of this result. Rather than a fixed network of *players*, we consider a random network of players, which induces a fixed interaction structure for the players' *types*, and the situation we consider is the following. Suppose that there is a set of types with small prior probability for whom conditional beliefs are very different under two priors (so that they may follow different strategies under the two priors). We analyze under what conditions these types do not "infect" a large (in terms of ex ante probability) set of types through players' higher order beliefs. This is the case precisely when there is a group of types with high prior probability which is sufficiently cohesive. Note that contagion is not physical in the current setting: players do not need even be in the same network to be affected by each other's behavior; rather, it is the correlation among players' types that leads—or does not lead—to contagion. This relation between our results and those of Morris (2000) shows that we can use the formal relation between network games with *complete* information and incomplete information games identified by Morris (1997, 2000) to study network games with *incomplete* information by considering the fixed network formed by types and their correlations that is induced by the random network of players, which is a result of independent interest.

The work in this chapter is related to three distinct literatures. Firstly, it is related to the literature on Bayesian network games (e.g. Galeotti et al., 2006; Jackson and Yariv, 2007; Sundararajan, 2005, also see Chapter 3). In Bayesian network games, the size of the network is commonly known. Moreover, it is often assumed that players' types are (asymptotically) independent.[2] By contrast, the class of network games of incomplete information we introduce in the current chapter allows for uncertainty about the network size, and for arbitrary correlations among player types.

Allowing for uncertainty about the network size and for correlation among player types is both important and natural. It is important because, as we have argued above, the assumptions on players' beliefs about the network size and the correlations among player types can have a qualitative effect on game-theoretic predictions. It is natural because agents will often be uncertain about the extent of their networks, and may well believe the types of other players to be correlated. As for uncertainty on the network size, the observation of Myerson (1998) that in some contexts, it is reasonable to assume that players are uncertain about the number of other players in the game holds *a fortiori* for network games, as in these games, players only interact with a small subset of players and have no direct information about the players they do not interact with. As for players' beliefs on the correlation among player types, there is ample evidence that many social and economic networks display positive assortativity, meaning that agents with a high (low) degree tend to be linked primarily with agents with a high (low) degree (see Jackson, 2008, and references therein). Moreover, evidence from social psychology suggests that individuals believe their networks to be highly clustered, i.e., that their networks contain a large number of small cycles (e.g. Crockett, 1982; Krackhardt and Kilduff, 1999). Hence, it is natural to assume that players believe that there is nonzero correlation among neighbor types.

The second literature to which the current work is related is the literature on games with population uncertainty. Games with population uncertainty in which all players interact directly have been studied by a number of authors (e.g. Kalai, 2004; McAfee and McMillan, 1987; Milchtaich, 2004; Myerson, 1998). In these games, players do not know how many players they interact with. By contrast, we consider a setting in which players interact locally, i.e., they only interact directly with a subset of players, and in which each player knows the number of players he interacts with. However, a player does not know the number of players his *neighbors* interact with. Hence, population uncertainty plays a distinctly different role here than in games with global interactions.

Finally, the current work builds on a literature that relates higher order beliefs to the equilibria of incomplete information games, in particular Monderer and

2 Galeotti et al. (2006) is a notable exception.

Samet (1989) and Kajii and Morris (1998), and we use extensively concepts and techniques from this literature. Kajii and Morris (1998) study lower hemicontinuity of the approximate equilibrium correspondence in Bayesian games with a (fixed) finite player set and a countably infinite state space.[3] They show that two priors over this state space are strategically close if and only if the prior probabilities of events are similar under the two priors and with high probability, it is approximate common knowledge that all players attach similar conditional probabilities to all events, i.e., with high probability, each player believes with high conditional probability that the conditional beliefs of all players are similar under the two priors and that all players believe with high conditional probability that the conditional beliefs of all players are similar, and that all players believe with high conditional probability that all players believe with high conditional probability... that the conditional beliefs of all players are similar under the two priors (for any number of iterations). Our result can thus be seen as a "spatial" analogue of this result: rather than requiring that all players believe that all players believe... that the conditional beliefs of all players are similar, we require that a player believes that his neighbors believe that their neighbors believe... that the conditional beliefs of their neighbors are similar.

Although we study the same issues as Kajii and Morris (1998), and follow their line of argument in our proofs,[4] conceptually, there are marked differences. We introduce the *local p-belief operator*, a belief operator in the sense of Monderer and Samet (1989). The local p-belief operator associates with each set of types a set of types that with conditional probability at least p interact exclusively with types in that set. It thus provides a measure of the "cohesiveness" of a set of types. We show that this operator quantifies players' higher order beliefs regarding local events in network games, i.e., a player's beliefs about his neighbors' beliefs about their neighbors' beliefs, and so on. The local p-belief operator is closely related to the p-belief operator of Monderer and Samet (1989), which quantifies players' higher order beliefs in Bayesian games. While the p-belief operator of Monderer and Samet (1989) can be used to characterize players' higher order beliefs over the global structure of the network, the local p-belief operator is well suited to characterize players' higher order beliefs over local events. Indeed, we show in Appendix 4.A that the local p-belief operator and the p-belief operator of Monderer and Samet (1989) (extended to the context of network games of incomplete information) are

3 Monderer and Samet (1996) study the related question under what conditions two information partitions are close in a strategic sense. That is, they fix the distribution over the states and vary players' information partitions. Milgrom and Weber (1985) study upper hemicontinuity of the Bayesian equilibrium correspondence.

4 Also see Rothschild (2005).

complementary in this respect.

The local p-belief operator is also related to the neighborhood operator of Morris (1997, 2000). Morris (1997) introduces the neighborhood operator in the context of games on a *fixed* network. For a given network, the neighborhood operator assigns to each subset of players the set of players in that subset for whom at least proportion p of their interactions is only with players in that subset. That is, the neighborhood operator relates to the cohesiveness of a group of players, just like the local p-belief operator relates to the cohesiveness of a set of types. Hence, the local p-belief operator shares features of both the p-belief operator of Monderer and Samet (1989) and the neighborhood operator of Morris (1997, 2000). Like the p-belief operator, the local p-belief operator pertains to players' (higher order) beliefs in incomplete information games. Like the neighborhood operator, the local p-belief operator refers to the local interactions of players.

This chapter is organized as follows. Preliminaries are discussed in Section 4.2. In Section 4.3, we introduce the class of network games of incomplete information. The local p-belief operator and players' higher order beliefs in network games are discussed in Section 4.4. Section 4.5 contains our main result and a discussion of its implications. Section 4.6 concludes. Appendix 4.A relates the local p-belief operator we introduce to the p-belief operator of Monderer and Samet (1989). Appendix 4.B contains the proofs that are not included in the main text.

4.2 Preliminaries

We assume that players are located on a network. Networks and random network models were introduced in Section 2.3; we briefly recall the most important definitions here for ease of reference. A *network* g is a pair consisting of a finite, nonempty set $V(g)$ of *vertices* and a finite set $E(g)$ of *edges*, with an edge being an unordered pair of two distinct vertices. Let g be a network. If $\{v, w\} \in E(g)$, where $v, w \in V, v \neq w$, then v and w are *neighbors* in g. For ease of notation, an edge $\{v, w\} \in E(g)$ is sometimes denoted by vw.

We consider a setting where the network is drawn from a class of networks according to some probability measure. Let $n \in \mathbb{N}$, and let $V^{(n)} := \{1, \ldots, n\}$. Let $\mathcal{G}^{(n)}$ be the set of all networks with vertex set $V^{(n)}$ and let

$$\mathcal{G} := \bigcup_{n \in \mathbb{N}} \mathcal{G}^{(n)}$$

be the countable set of all networks with a finite vertex set. Define

$$\mathcal{V} := \bigcup_{n \in \mathbb{N}} V^{(n)} = \mathbb{N}.$$

Let $\mathscr{F}$ be the σ-algebra generated by the set of singletons of $\mathcal{G}$. Let $\mathcal{M}$ denote the set of all probability measures on $(\mathcal{G}, \mathscr{F})$, and let $\mu \in \mathcal{M}$. The probability space $(\mathcal{G}, \mathscr{F}, \mu)$ is a *random network model*. Example 4.2.1 gives a simple example of a random network model with a random number of vertices; for a particularly elegant model of a random network with a random number of vertices, see Bollobás et al. (2007).

Example 4.2.1. Suppose that a population evolves in (discrete) generations, indexed by $m \in \{0, 1, \ldots\}$. Each member of the mth-generation gives birth to a family (possibly empty) of members of the $(m+1)$th generation. The number of offspring that each individual produces is a random variable, and is independent of the number of offspring of all other individuals. The distribution of the number of offspring is the same for each individual. This is a simple branching process (e.g. Grimmett and Stirzaker, 1992). If we associate a vertex with each individual and if we interpret ancestry relations as (undirected) edges, this random process gives rise to a network with a random number of vertices. ◃

In the current framework, we associate a player with each vertex, so that edges represent the relations between players. In the following, we therefore refer to players rather than to vertices. Furthermore, random network models represent players' beliefs. Throughout this chapter, we therefore refer to a random network model as a *network belief system*.

We are interested in the local environment of vertices. Let $g \in \mathcal{G}$, and let $i \in V(g)$. Then, $N_i(g)$ is the set of neighbors of player i in g, i.e., his *neighborhood* in g. The number of neighbors of i in g is his *degree* $D_i(g)$ in g. We are also interested in the number of neighbors the neighbors of a given vertex have. Loosely speaking (see Section 2.3 for a precise definition), the *neighbor degree profile* $K_i(g)$ of player i in g is a list of the degrees of the neighbors of the vertex, in a non-increasing order. For $t \in \mathbb{N}$, Ω_K^t denotes the set of all neighbor degree profiles for a player of degree t, and we define $\Omega_K := \bigcup_t \Omega_K^t$. Finally, let $\mathscr{F}_K$ be the σ-field generated by the set of singletons of Ω_K.

The following definition will be useful when specifying players' beliefs in the next section. Let $n \in \mathbb{N}$. Two networks $g, g' \in \mathcal{G}^{(n)}$ are *isomorphic* if there is a permutation π of $V^{(n)}$ such that $\{i, j\} \in E(g)$ for $i, j \in V^{(n)}, i \neq j$, if and only if

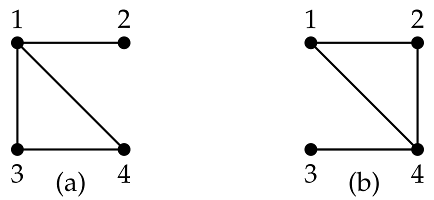


Figure 4.1. Two isomorphic networks. To see that these networks are isomorphic, notice that there are two permutations of the vertex set $V^{(4)} = \{1, 2, 3, 4\}$ that renders the network under (a) into the network under (b): (*i*) $\pi(i) = 5 - i$ for each $i \in V^{(4)}$, (*ii*) $\pi'(1) = 4, \pi'(2) = 3, \pi'(3) = 1$, $\pi'(4) = 2$.

$\{\pi(i), \pi(j)\} \in E(g')$. This defines an equivalence relation; hence, the set $\mathcal{G}^{(n)}$ can be partitioned into a finite number of *isomorphism classes*, i.e., sets of isomorphic networks. Let $\mathscr{C}^{(n)}$ be the collection of isomorphism classes of $\mathcal{G}^{(n)}$, and let $\mathscr{C} := \bigcup_{n \in \mathbb{N}} \mathscr{C}^{(n)}$ be the collection of isomorphism classes of $\mathcal{G}$. Figure 4.1 depicts two networks that are isomorphic.

Throughout this chapter, we make the following two assumptions on network belief systems:

Assumption 4.A (Finite expected number of vertices). The network belief system $(\mathcal{G}, \mathscr{F}, \mu)$ is such that the expected number of vertices is finite, i.e.,

$$\sum_{n \in \mathbb{N}} n\mu\left(\mathcal{G}^{(n)}\right) < \infty. \qquad \triangleleft$$

Assumption 4.B (No isolated vertices). The network belief system $(\mathcal{G}, \mathscr{F}, \mu)$ is such that with probability 1, each vertex has at least one neighbor. That is,

$$\mu(\{g \in \mathcal{G} \mid D_i(g) > 0 \text{ for all } i \in V(g)\}) = 1. \qquad \triangleleft$$

Assumption 4.B is for notational convenience only and can easily be relaxed.

4.3 Network games of incomplete information

4.3.1 Game

A network game of incomplete information is a game on a network, in which players are associated with a vertex in the network, and each player's payoff depends on the types and actions of himself and his neighbors. Players have incomplete information on the network: they have a common prior over the class $\mathcal{G}$ of all finite networks, and they know the number of neighbors they have, i.e., their degree. In particular, they may not know the number of players in the network.

Formally, let $(\mathcal{G}, \mathscr{F}, \mu)$ be a network belief system satisfying Assumptions 4.A and 4.B. A network $g \in \mathcal{G}$ is drawn according to $(\mathcal{G}, \mathscr{F}, \mu)$. Each vertex in the set $V(g)$ represents a player, and we refer to a player by his vertex label. Players do not know their vertex label, however.[5] Each player $i \in V(g)$ knows the number of neighbors he has in the network: his *type* is his degree. Hence, the *type set* is $T = \mathbb{N}_0$. Henceforth, we will speak of type and *neighbor type profile*, rather than of degree and neighbor degree profile. Each player is endowed with a finite, nonempty set A of pure strategies or *actions*. For each $t \in T$, the *payoffs* of a player of type t are given by a function v_t. For $t = 0$, v_t is a real function on A, i.e., the payoffs to an isolated player only depend on his own action. For $t > 0$, v_t is a function from $A \times A^t \times T^t$ to $\mathbb{R}$ that is symmetric in A^t and T^t, i.e., for all permutations π on $\{1, \ldots, t\}$, for all $a \in A, (a_1, \ldots, a_t) \in A^t, (\theta_1, \ldots, \theta_t) \in T^t$,

$$v_t\Big(a, (a_1, \ldots, a_t), (\theta_1, \ldots, \theta_t)\Big) = v_t\Big(a, (a_{\pi(1)}, \ldots, a_{\pi(t)}), (\theta_{\pi(1)}, \ldots, \theta_{\pi(t)}\Big),$$

with $v_t\Big(a, (a_1, \ldots, a_t\Big), \Big(\theta_1, \ldots, \theta_t)\Big)$ the payoffs to a player of type t with neighbor type profile $(\theta_1, \ldots, \theta_t)$ when he chooses action $a \in A$, and his neighbors play according to the action profile $(a_1, \ldots, a_t)$.

Definition 4.3.1. *A* network game of incomplete information *is a tuple*

$$\langle T, A, (\mathcal{G}, \mathscr{F}, \mu), (v_t)_{t \in T} \rangle$$

with its elements defined as above.

We fix the action set A. A network game of incomplete information is then fully characterized by the common prior on $(\mathcal{G}, \mathscr{F})$ and its profile of payoff functions.

[5] The vertex labelling is introduced merely to be able to define random variables such as the degree of vertices. However, the labelling is completely arbitrary and carries no meaning.

A network game of incomplete information $\langle T, A, (\mathcal{G}, \mathscr{F}, \mu), (v_t)_{t\in T}\rangle$ is henceforth denoted by the pair (μ, v), where $v := (v_t)_{t\in T}$.

Let $B \in \mathbb{R}$. A profile v of payoff functions is *bounded by B* if for all $t \in T, t \neq 0$, $\theta \in \Omega_K^t$ and for all $a, a' \in A^{t+1}$,

$$\max\Big\{|v_t(a,\theta) - v_t(a',\theta)|, |v_t(a,\theta)|\Big\} \leq B.$$

If there exists $B \in \mathbb{R}$ such that the profile v is bounded by B, we say that it is *bounded*.

As in games with population uncertainty and random-player games (Myerson, 1998; Milchtaich, 2004), the player set is not commonly known, so that players are not aware of the particular identities of the other players in the game. Hence, we cannot assign a separate strategy to each individual player. Rather, a strategy can only depend on a player's type. Hence, for each type $t \in T$, let σ_t be a real function defined on A which satisfies

$$\sigma_t(a) \geq 0$$

for all $a \in A$, and

$$\sum_{a\in A} \sigma_t(a) = 1,$$

with $\sigma_t(a)$ the probability that a player of type t chooses action a. The set of all distributions on A is denoted by Σ. An element $\sigma = (\sigma_0, \sigma_1, \sigma_2, \ldots) \in \Sigma^T$ is referred to as a *strategy function*.

4.3.2 Beliefs

To calculate expected payoffs, we need to specify players' beliefs. There are two issues to note. Firstly, as in games with population uncertainty and random-player games, players condition on their type, as well as on the fact that they are selected to play. That is, from a player's perspective, even if all networks in the support of μ have equal probability ex ante, he believes that he is more likely to belong to a network with many players: there are simply more vertices to be associated with in large networks (cf. Myerson, 1998; Milchtaich, 2004). This is illustrated in Example 4.3.2.

Secondly, a player cannot distinguish between networks in a given isomorphism class, as he does not know his vertex label or the vertex labels of his opponents. Hence, to calculate players' beliefs that they have a given neighbor type

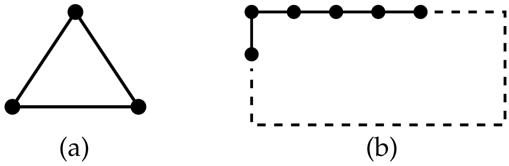


Figure 4.2. The networks of Example 4.3.2. (a) The network $g^{(3)}$; (b) The network $g^{(300)}$.

profile, we need to consider the probability measure on the collection of isomorphism classes induced by μ, and for each isomorphism class, we need to take into account the number of vertices with that neighbor type profile in the isomorphism class.

Example 4.3.2. Suppose that the network belief system assigns probability $\frac{1}{2}$ to the network $g^{(3)}$ consisting of a triangle of three players, and probability $\frac{1}{2}$ to the network $g^{(300)}$ consisting of 300 players, connected in a cycle (see Figure 4.2). Though the prior probability of the two networks is $\frac{1}{2}$, from the perspective of a player, it is much more likely that network $g^{(300)}$ is realized, as to each "player position" in $g^{(3)}$, there are 100 player positions in $g^{(300)}$. Using Bayes' rule, a player's belief that $g^{(300)}$ is realized, given that he is a player in the network, is

$$\frac{300 \cdot \frac{1}{2}}{3 \cdot \frac{1}{2} + 300 \cdot \frac{1}{2}} = \frac{300}{303}. \qquad \triangleleft$$

Formally, recall that $\mathscr{C}$ is the collection of isomorphism classes of $\mathcal{G}$, and that $\mathscr{F}_K$ is the σ-field associated with the set of all neighbor type profiles Ω_K. For each $C \in \mathscr{C}$, and each $F \in \mathscr{F}_K$, let $n_C(F)$ be the number of vertices in a network in C with their neighbor type profile in F. Note that $n_C(F)$ is well defined: for any two networks $g, g' \in C$, the number of vertices with their neighbor type profile in F is identical. Let

$$\hat{n} := \sum_{n \in \mathbb{N}} n\mu\left(\mathcal{G}^{(n)}\right)$$

be the expected number of players in the network belief system. By Assumption 4.A, $\hat{n}$ is finite. Consider a player who is called upon to play, but who does not know his type yet. The probability that the neighbor type profile of such a player lies in the set F is

$$q_\mu(F) = \frac{1}{\hat{n}} \sum_{C \in \mathscr{C}} \mu(C)\, n_C(F),$$

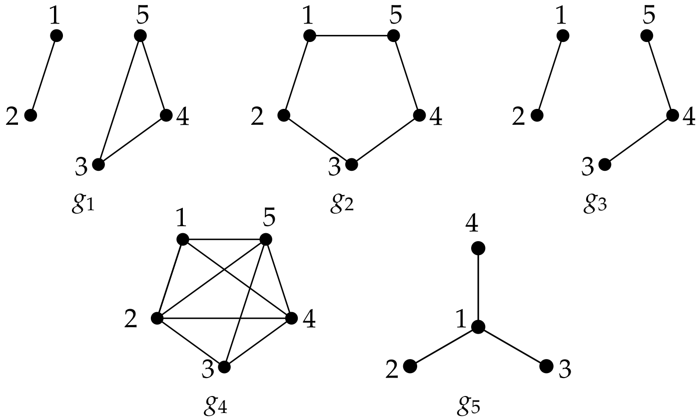


Figure 4.3. The networks representing the isomorphism classes of Example 4.3.3 that have positive probability.

where we recall that $\mu(C)$ is the prior probability that a network from the isomorphism class C is realized. In words, $q_\mu(F)$ is equal to the expected fraction of players with a neighbor type profile in F. We refer to $q_\mu(F)$ as the *prior probability* that a player's neighbor type profile is in F. In particular, for each $t \in T$,

$$q_\mu(t) := q_\mu(\Omega_K^t)$$

denotes the prior probability that a player's type is t. It can be readily checked from the definitions that q_μ is indeed a probability measure on the measurable space $(\Omega_K, \mathscr{F}_K)$ of neighbor type profiles:

(a) $q_\mu(\emptyset) = 0$, and $q_\mu(\Omega_K) = 1$;
(b) q_μ satisfies σ-additivity: for $A_1, A_2, \ldots$ a collection of disjoint members of $\mathscr{F}_K$,

$$q_\mu\Big(\bigcup_{k \in \mathbb{N}} A_k\Big) = \sum_{k \in \mathbb{N}} q_\mu(A_k).$$

Example 4.3.3 illustrates the calculation of players' beliefs.

Example 4.3.3. Suppose that a network belief system assigns positive probability only to the networks $g_1, g_2, \ldots, g_5$ in Figure 4.3 or to networks isomorphic to them. Suppose that all isomorphism classes associated with the networks in Figure 4.3 have equal probability, i.e., for each isomorphism class $C \in \mathscr{C}$ of $\mathcal{G}$, $\mu(C) = \frac{1}{5}$ if there is a network $g \in \{g_1, g_2, \ldots, g_5\}$ such that $g \in C$, and $\mu(C) = 0$ otherwise.

To calculate a player's prior belief that his neighbor type profile is in some set $F \in \mathscr{F}_K$, we now simply need to count the number of vertices in $g_1, \dots, g_5$ with their neighbor type profile in F, and compare this to the total number of vertices in $g_1, \dots, g_5$. For instance, a player's prior belief that his type is $t = 2$ is

$$q_\mu(t) = \frac{\frac{1}{5} \cdot 3 + \frac{1}{5} \cdot 3 + \frac{1}{5} \cdot 1}{4 \cdot \frac{1}{5} \cdot 5 + \frac{1}{5} \cdot 4} = \frac{9}{24},$$

and a player's prior belief that his neighbor type profile is $\theta = (2,2)$ is $q_\mu(\theta) = 8/24$. This is intuitive: there are 24 vertices in total in the networks $g_1, \dots, g_5$, of which 9 vertices have type $t = 2$ and 8 vertices have neighbor type profile $(2,2)$. Noting that from a player's perspective, he is equally likely to be associated with any of the vertices in $g_1, \dots, g_5$, we obtain the values above. ◃

Conditional probabilities can be calculated in the usual way. Let $t \in T$ be such that $q_\mu(t) > 0$. A player's belief that his neighbor type profile is in the set $F \in \mathscr{F}_K$ given that his type is t is given by

$$\begin{aligned} q_\mu(F \mid t) &:= \frac{q_\mu(F \cap \Omega_K^t)}{q_\mu(\Omega_K^t)} \\ &= \frac{\sum_{C \in \mathscr{C}} \mu(C) n_C(F \cap \Omega_K^t)}{\sum_{C \in \mathscr{C}} \mu(C) n_C(\Omega_K^t)}. \end{aligned}$$

With minor abuse of notation, we write $q_\mu(\theta \mid t)$ to denote $q_\mu(\{\theta\} \mid t)$ for $\theta \in \Omega_K$. We refer to $q_\mu(F \mid t)$ as the *conditional belief* of (a player of) type t that his neighbor type profile is in F.

Example 4.3.3 (continued). In order to calculate a player's conditional belief that his neighbor type profile is in some set $F \in \mathscr{F}_K$ given that his type is $t \in T$, we need to count the number of vertices in $g_1, \dots, g_5$ with type t and neighbor type profile in F, and compare this to the total number of vertices in $g_1, \dots, g_5$ with type t. For instance, a player's conditional belief that his neighbor type profile is $\theta = (2,2)$ given that his type is $t = 2$ is

$$q_\mu(\theta \mid t) = \frac{\frac{1}{5} \cdot 5 + \frac{1}{5} \cdot 3}{\frac{1}{5} \cdot 5 + \frac{1}{5} \cdot 3 + \frac{1}{5} \cdot 1} = \frac{8}{9}.$$

Indeed, eight out of the nine vertices in $g_1, \dots, g_5$ with type $t = 2$ have neighbor type profile $\theta = (2,2)$. ◃

Remark 4.3.4. Tacitly we have assumed that there is some pool of candidate players from which (actual) players are drawn. We have not specified this pool, nor have we specified the method by which players are selected. There is no need to specify this, however, as we are solely interested in players' beliefs *given* that they have been selected to play. Hence, the probability measure q_μ gives the probability that an arbitrary player has a certain neighbor type profile. Also see Myerson (1998, pp. 382–384) on this point. ◃

4.3.3 Payoffs and equilibrium

Now that we have calculated players' beliefs, we can define expected payoffs. Let $t \in T, t \neq 0$, $\theta = (\theta_1, \ldots, \theta_t) \in \Omega_K^t$, and define $\sigma_{(\theta)} := (\sigma_{\theta_1}, \ldots, \sigma_{\theta_t}) \in \Sigma^t$. Let

$$v_t(a, \sigma_{(\theta)}, \theta) := \sum_{a^{(t)} \in A^t} \left(\prod_{\ell=1}^{t} \sigma_{\theta_\ell}\left(a_\ell^{(t)}\right) \right) v_t\left(a, a^{(t)}, \theta\right).$$

For each type $t \in T$ such that $q_\mu(t) > 0$, the *expected payoffs* to a player of type t of an action $a \in A$ when the other players play according to the strategy function $\sigma \in \Sigma^T$ are

$$\varphi_t(a, \sigma; \mu) := \sum_{\theta \in \Omega_K^t} q_\mu(\theta \mid t)\, v_t(a, \sigma_{(\theta)}, \theta). \tag{4.1}$$

For $t \in T$ such that $q_\mu(t) = 0$, set $\varphi_t(a, \sigma; \mu) := 0$ for all $a \in A$ and $\sigma \in \Sigma^T$. Also, for each $t \in T$ and $\sigma \in \Sigma^T$, let

$$\varphi_t(\sigma; \mu) := \sum_{a \in A} \sigma_t(a) \varphi_t(a, \sigma; \mu). \tag{4.2}$$

The *type-averaged (expected) payoffs* of strategy function $\sigma \in \Sigma^T$ are

$$\Phi(\sigma; \mu) := \sum_{t \in T} q_\mu(t) \varphi_t(\sigma; \mu). \tag{4.3}$$

The type-averaged payoff of a strategy function $\sigma \in \Sigma^T$ is the weighted average of the expected payoffs of the different types under the strategy function σ, and gives the expected payoff of a player who is called upon to play the game, but does not know his type yet. Hence, the expected payoffs of a type correspond to the interim expected payoffs of a player in standard Bayesian games, while the type-averaged payoffs correspond to the ex ante expected payoffs in Bayesian games.

Definition 4.3.5. *Let $\varepsilon \geq 0$. A strategy function $\sigma \in \Sigma^T$ is an ε-*equilibrium *of a network game of incomplete information (μ, v) if for each $t \in T$ such that $q_\mu(t) > 0$, for each action $a \in A$ such that $\sigma_t(a) > 0$,*

$$\varphi_t(a, \sigma; \mu) \geq \varphi_t(b, \sigma; \mu) - \varepsilon$$

for all $b \in A$. We refer to a 0*-equilibrium as an* equilibrium.

Proposition 4.3.6. *Let (μ, v) be a network game of incomplete information. If the profile of payoff functions v is bounded, the game has an equilibrium.*

Proof. See Appendix 4.B. □

Let (μ, v) be a network game of incomplete information. Then, $\mathcal{N}^{\varepsilon}(\mu, v)$ denotes the set of ε-equilibria of (μ, v). In particular, $\mathcal{N}^{0}(\mu, v)$ denotes the set of equilibria of (μ, v).

4.4 The local belief operator and higher order beliefs

To answer the question under which conditions two priors are close in a strategic sense in network games of incomplete information, we need some tools to quantify players' (higher order) beliefs. In this section, we develop these tools, which we will use in the next section to address the question of strategic closeness.

Let $\mu \in \mathcal{M}$, and let $p \in [0,1]$. The *local p-belief operator* B^p_μ associates with each set of types the subset of types that with conditional probability at least p interact exclusively with types in that set (whenever they have positive probability). Formally, let $S \subseteq T$. Then,

$$B^p_\mu(S) := \{t \in S \mid q_\mu(t) > 0 \Rightarrow q_\mu(S^t \mid t) \geq p\}. \tag{4.4}$$

Note that $B^p_\mu(S)$ includes the types in S that have zero probability. By definition, $B^p_\mu(S) \subseteq S$. If also

$$B^p_\mu(S) \supseteq S, \tag{4.5}$$

we say that the set of types S is *p-closed* (under μ).[6] If a set of types is p-closed, then each type in the set interacts with high conditional probability only with types in that set, who in turn interact with high conditional probability only with types in that set, and so on.

The local p-belief operator can be iterated any finite number of times. For instance, $B^p_\mu\big(B^p_\mu(S)\big)$ is the set of types $t \in B^p_\mu(S)$ such that with conditional probability

6 We follow the convention in the literature on higher order beliefs of making the one-sided implications explicit, as it is the one-sided implication in (4.5) that captures the nature of a set being p-closed.

at least p, they interact exclusively with types in $B^p_\mu(S)$, that is, with types in S that with conditional probability at least p interact exclusively with types in S. Define

$$\left[B^p_\mu\right]^1(S) := B^p_\mu(S),$$

and, for each $\ell \in \mathbb{N}$, let

$$\left[B^p_\mu\right]^{\ell+1} = B^p_\mu \circ \left[B^p_\mu\right]^\ell.$$

Let

$$C^p_\mu(S) := \bigcap_{\ell\in\mathbb{N}} \left[B^p_\mu\right]^\ell(S)$$

be the set of types that with conditional probability at least p interact exclusively with types that with conditional probability at least p... interact exclusively with types in S, for any number of iterations.

Example 4.3.3 (continued). Let $S := \{1,2,3\}$. It is easy to check that the conditional belief of a player with type $t = 1$ or $t = 2$ that he interacts exclusively with players with types in S is $q_\mu(S^t \mid t) = 1$, while the conditional belief of a player with type $t = 3$ that he interacts exclusively with players with types in S is $q_\mu\big(S^3 \mid 3\big) = \frac{1}{3}$. Hence, for $p \in [0, \frac{1}{3}]$, we have $B^p_\mu(S) = S$, while for $p \in (\frac{1}{3}, 1]$, it holds that $B^p_\mu(S) = \{1,2\}$. Now consider the conditional beliefs of players with types in the set $B^p_\mu(S)$ that they only interact with players with a type in $B^p_\mu(S)$. For instance, for $p \in (\frac{1}{3}, 1]$, it can be easily verified that

$$q_\mu\Big(B^p_\mu(S) \mid 1\Big) = \tfrac{2}{3},$$

while

$$q_\mu\Big((B^p_\mu(S))^2 \mid 2\Big) = 1.$$

Hence, for $p \in (\frac{1}{3}, \frac{2}{3})$, it holds that $B^p_\mu(B^p_\mu(S)) = \{1,2\}$, while for $p \in (\frac{2}{3}, 1]$, we have $B^p_\mu(B^p_\mu(S)) = \{2\}$. ◁

The local p-belief operator satisfies the following desirable properties:[7]

Monotonicity: For any $T', T'' \subseteq T$, if $T' \subseteq T''$, then

$$B^p_\mu(T') \subseteq B^p_\mu(T'').$$

7 See Monderer and Samet (1989, 1996) for a discussion of these axioms. Note that the axiom of monotonicity implies the axiom of subpotency in the current context: for all $S \subseteq T$, $B^p_\mu\big(B^p_\mu(S)\big) \subseteq B^p_\mu(S)$.

Continuity: Let $S \subseteq T$, and for $k \in \mathbb{N}$, let $T_k \subseteq T$. If $T_k \downarrow S$, i.e., if $(T_k)_{k \in \mathbb{N}}$ is a (weakly) decreasing sequence and $\bigcap_{k \in \mathbb{N}} T_k = S$, then

$$B^p_\mu(T_k) \downarrow B^p_\mu(S).$$

Continuity in *p*: If $p_k \uparrow p$, then, for any $S \subseteq T$,

$$B^{p_k}_\mu(S) \downarrow B^p_\mu(S).$$

For proofs, see Appendix 4.B.

The following two results, which we will use later on, have well-known counterparts in the literature on higher order beliefs (Monderer and Samet, 1989, Prop. 3).

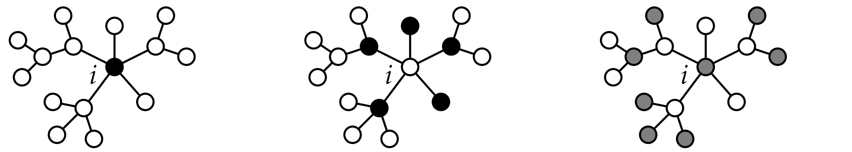
Lemma 4.4.1. *Let $S \subseteq T$, and let $p \in [0,1]$. The set of types $C^p_\mu(S)$ is p-closed, i.e.,*

$$B^p_\mu\big(C^p_\mu(S)\big) = C^p_\mu(S).$$

Lemma 4.4.2. *Let $p \in [0,1]$. Let $t \in T$, and let $T' \subseteq T$. We have that $t \in C^p_\mu(T')$ if and only if there exists a subset of types $S \subseteq T'$ that is p-closed such that $t \in S$ and $S \subseteq B^p_\mu(T')$.*

The proofs of Lemmas 4.4.1 and 4.4.2 can be found in Appendix 4.B.

Though at first sight the local p-belief operator seems to refer primarily to the "cohesiveness" of a set of types, we can use the local p-belief operator to characterize players' higher order beliefs, i.e., the beliefs players have over the beliefs of other players over the beliefs of other players, and so on. For instance, consider the set $B^p_\mu(S)$ of types for some $S \subseteq T$. We have said that with conditional probability at least p, a player with type $t \in B^p_\mu(S)$ interacts exclusively with players whose types lie in S. An alternative formulation is that a player with a type $t \in B^p_\mu(S)$ believes, given his type, that with probability at least p, all his neighbors have their types in the set S. When the local p-belief operator is iterated, we obtain statements about players' higher order beliefs. When $t \in B^p_\mu(B^p_\mu(S))$, a player with type t believes (with conditional probability at least p) that his neighbors believe that their neighbors' types are in S (see Figure 4.4(a)). Similarly, when $t \in B^p_\mu(B^p_\mu(B^p_\mu(S)))$, a player believes that his neighbors believe that their neighbors believe that their neighbors' types are in S (see Figure 4.4(b)). That is, the local p-belief operator is a belief operator in the sense of Monderer and Samet (1989) restricted to events of the form "the types of all neighbors of an arbitrary player are in a given set". We discuss the relation between the local p-belief operator and the p-belief operator more extensively in Appendix 4.A.



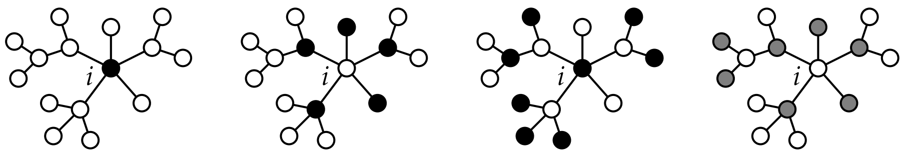


Figure 4.4. Higher order beliefs in a network. (a) Suppose player i has a type in $B^p_\mu\big(B^p_\mu(S)\big)$. Then, with conditional probability at least p, he believes that his neighbors have a type in S, and that with conditional probability at least p, they believe that their neighbors' types lie in S. (b) Suppose player i has a type in $B^p_\mu\Big(B^p_\mu\big(B^p_\mu(S)\big)\Big)$. Then, with conditional probability at least p, he believes that his neighbors have a type in S, and that with conditional probability at least p, they believe that their neighbors have a type in S, and that with conditional probability at least p, they believe that their neighbors' types lie in S.

The local p-belief operator also allows us to characterize a player's beliefs over others' beliefs about himself and his beliefs. Indeed, a player is a neighbor of his neighbors, so that when a player believes (with high conditional probability) that his neighbors believe that their neighbors' types are in S (i.e., a player's type is in $B^p_\mu(B^p_\mu(S))$), then he believes that the players he interacts with believe that his type is in S. Similarly, if a player believes that his neighbors believe that their neighbors believe that their neighbors' types are in S (i.e., a player's type is in $B^p_\mu(B^p_\mu(B^p_\mu(S))))$, then he believes that his neighbors believe that he believes that their types are in S.

We will use the local p-belief operator extensively in the next section to analyze players' beliefs in network games of incomplete information.

4.5 Strategic convergence

4.5.1 Main result

We want to quantify the extent to which priors are similar in a strategic sense. To that aim, we define a measure on the set of priors such that if two priors are close according to this measure, then, for each network game of incomplete information, for each equilibrium of the game in which beliefs are given by one of these priors, there exists an approximate equilibrium of the game with the other prior, such that type-averaged payoffs are close in both equilibria. If that is the case, then, for each possible profile of payoff functions, each player who is called upon to play can obtain approximately the same payoffs (in an ex ante sense) under both priors: from a player's (ex ante) perspective, the two priors are similar. We want to find the weakest conditions that guarantees that the above holds.

Formally, let $\mu, \mu' \in \mathcal{M}$, and let $v := (v_t)_{t \in T}$ be a profile of payoff functions. For each $\varepsilon \geq 0$, define

$$\chi(\mu, \mu'; v, \varepsilon) := \sup_{\sigma \in \mathcal{N}^0(\mu, v)} \inf_{\sigma' \in \mathcal{N}^\varepsilon(\mu', v)} |\Phi(\sigma; \mu) - \Phi(\sigma'; \mu')|,$$

where Φ is the type-averaged payoff given profile v of payoff functions. That is, for a given $\varepsilon \geq 0$, for each equilibrium under μ, we first find an ε-equilibrium under μ' which minimizes the (absolute) difference in type-averaged payoffs under both equilibria, and we then look for the equilibrium under μ which maximizes this difference. This formalizes the idea that for *each* equilibrium of the network game of incomplete information with one prior, there exists *some* approximate equilibrium of the network game of incomplete information with the other prior, such that type-averaged payoffs are similar under both equilibria. To obtain a symmetric function of μ and μ', let

$$\chi^*(\mu, \mu'; v, \varepsilon) := \max\{\chi(\mu, \mu'; v, \varepsilon), \chi(\mu', \mu; v, \varepsilon)\}.$$

We refer to $\chi^*(\mu, \mu'; v, \varepsilon)$ as the *strategic distance* between μ and μ' for the profile v given ε. The supremum of $\chi^*(\mu, \mu'; v, \varepsilon)$ over profiles v that are bounded is called the *strategic distance* between μ and μ' given ε.

Clearly, when ε increases, the set of ε-equilibria weakly increases, as more and more strategies will satisfy the equilibrium criterion, so that the (absolute) difference in type-averaged expected payoffs will decrease weakly. Hence, we are interested in the strategic distance between priors given ε when ε comes arbitrarily close to 0. This leads us to the following definition (cf. Kajii and Morris, 1998):

Definition 4.5.1. *Let $\mu \in \mathcal{M}$, and consider a sequence $(\mu^k)_{k\in\mathbb{N}}$ in $\mathcal{M}$. The sequence $(\mu^k)_{k\in\mathbb{N}}$* converges strategically *to μ if for each profile v of payoff functions that is bounded, for each $\varepsilon > 0$, we have that*

$$\lim_{k\to\infty} \chi^*(\mu, \mu^k; v, \varepsilon) = 0.$$

A natural requirement for strategic convergence is that priors attach similar probabilities to the event that a player has a neighbor type profile in a certain set, i.e., that priors converge in the weak topology on Ω_K (cf. Chapter 3). Hence, define

$$d_0(\mu, \mu') := \sup_{F\in\mathscr{F}_K} |q_\mu(F) - q_{\mu'}(F)|. \tag{4.6}$$

We also need to consider players' conditional beliefs, i.e., the beliefs they have over their neighbors' types and beliefs, given their own type. For $\delta \in [0, 1]$, let

$$T^\delta_{\mu,\mu'} := \left\{ t \in T \;\middle|\; q_\mu(t) > 0, q_{\mu'}(t) > 0 \Rightarrow \sup_{F\in\mathscr{F}_K} |q_\mu(F \mid t) - q_{\mu'}(F \mid t)| \le \delta \right\} \tag{4.7}$$

be the set of types such that players' conditional beliefs on their neighbors' types are within δ, whenever the type has positive probability under μ and μ'. If δ is small, the conditional beliefs of a player with a type $t \in T^\delta_{\mu,\mu'}$ over the types of his neighbors are close under μ and μ'. If a player has a type $t \notin T^\delta_{\mu,\mu'}$, then his optimal strategy under μ and μ' may differ substantially, as he believes (given his type) that his local environment is very different under μ and μ'.

However, even if with high (prior) probability, a player has a type such that his conditional beliefs on his neighbors' types are similar under μ and μ' (i.e., that his type is in $T^\delta_{\mu,\mu'}$), outcomes can be very different under the two priors. The reason is that a player may believe with high conditional probability that the conditional beliefs of some of his neighbors on their neighbors' types are very different under μ and μ' (i.e., $t \notin B^p_\mu(T^\delta_{\mu,\mu'})$ for some $p \in [0, 1]$), or that some of his neighbors believe with high conditional probability that the conditional beliefs of some of their neighbors are very different under μ and μ' (i.e., $t \notin B^p_\mu\big(B^p_\mu(T^\delta_{\mu,\mu'})\big)$), and so on. Hence, we need to require that with high probability, a player has a type in the set $C^p_\mu(T^\delta_{\mu,\mu'})$, for some large $p \in [0, 1]$. In that case, a player's conditional beliefs are similar under μ and μ', and, he believes with high conditional probability that the conditional beliefs of his neighbors are similar under the two priors and that his neighbors believe with high conditional probability that the conditional beliefs of their neighbors are similar under the two priors, and so on. This makes that the actions that are optimal for a player of type $t \in C^p_\mu(T^\delta_{\mu,\mu'})$ under μ will be (almost) optimal under μ', as he expects his neighbors to behave similarly under μ and μ' (as his neighbors expect their neighbors to behave similarly, as the neighbors of his neighbors expect their neighbors...).

Formally, for each $S \subseteq T$, let

$$\Theta(S) := \bigcup_{t \in S} \Omega_K^t$$

be the set of neighbor type profiles in which the type of the "central" player belongs to the set S. Then, define

$$d_1(\mu, \mu') := \inf\Big\{\delta \in [0,1] \;\Big|\; q_\mu\Big(\Theta(C_\mu^{1-\delta}(T_{\mu,\mu'}^\delta))\Big) \geq 1 - \delta\Big\}. \tag{4.8}$$

If $d_1(\mu, \mu')$ is small, then, with high prior probability (under μ), a player has a type such that his conditional beliefs are similar under μ and μ', and with high conditional probability, he interacts exclusively with players whose conditional beliefs are close, and who, with high conditional probability, interact exclusively with players whose conditional beliefs are close, and so on.

Remark 4.5.2. One may think that requiring that with high prior probability, a player has a type in $C_\mu^p(T_{\mu,\mu'}^\delta)$ may not be sufficient. Even if a player believes, given his type, that with high probability his neighbors will choose the same actions under μ and μ' (allowing for ε-best responses), they may not do so if in fact their type is not in $C_\mu^p(T_{\mu,\mu'}^\delta)$. That is, if with high probability, some of the neighbors of a player have a type $t \notin C_\mu^p(T_{\mu,\mu'}^\delta)$, the payoff to a player with type $t \in C_\mu^p(T_{\mu,\mu'}^\delta)$ can be very different under μ and μ'.[8] However, Lemma 4.5.4 below shows that, if the probability is high that a player has a type in the set $C_\mu^p(T_{\mu,\mu'}^\delta)$, then in fact also the probability that his neighbors have a type in $C_\mu^p(T_{\mu,\mu'}^\delta)$ will be high. Hence, it is sufficient to require that with high probability, a player has a type in $C_\mu^p(T_{\mu,\mu'}^\delta)$. ◁

We can combine (4.6) and (4.8) to obtain

$$d^*(\mu, \mu') := \max\{d_0(\mu, \mu'), d_1(\mu, \mu'), d_1(\mu', \mu)\}. \tag{4.9}$$

It is immediate that d^* is nonnegative and symmetric. Moreover, $d^*(\mu, \mu') = 0$ if and only if $\mu = \mu'$. However, d^* need not satisfy the triangle inequality, so that it is not a metric. Yet, d^* generates a topology on the set $\mathcal{M}$ of probability measures on $(\mathcal{G}, \mathscr{F})$: a sequence $(\mu^k)_{k \in \mathbb{N}}$ converges to μ if and only if for any $\varepsilon > 0$, there exists $K_\varepsilon \in \mathbb{N}$ such that $d^*(\mu^k, \mu) \leq \varepsilon$ for all $k > K_\varepsilon$.

8 Indeed, Kajii and Morris (1998) require that the prior probability that *all* players have close conditional beliefs should be high.

We are now ready to state our main result.

Theorem 4.5.3. *Let $\mu \in \mathcal{M}$ and let $(\mu^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{M}$. Then, $(\mu^k)_{k\in\mathbb{N}}$ converges strategically to μ if and only if*

$$\lim_{k\to\infty} d^*(\mu, \mu^k) = 0.$$

Theorem 4.5.3 follows from Propositions 4.5.6–4.5.8. The proof of Proposition 4.5.6 uses Lemma 4.5.4 and Lemma 4.5.5.

Lemma 4.5.4. *Let $\mu \in \mathcal{M}$, and fix $\alpha, p \in [0,1]$. For each $S \subseteq T$, if the probability that a player has a type in the set $C^p_\mu(S)$ is at least α, i.e., if*

$$q_\mu\left(\Theta\left(C^p_\mu(S)\right)\right) \geq \alpha,$$

then the probability that this player and his neighbors have their types in $C^p_\mu(S)$ is at least αp:

$$q_\mu\Big(\bigcup_{t\in C^p_\mu(S)} \left(C^p_\mu(S)\right)^t\Big) \geq \alpha p.$$

Proof. See Appendix 4.B. □

Lemma 4.5.5. *Let $\mu, \mu' \in \mathcal{M}$, and let $\delta \in [0,1]$. Let v be a profile of payoff functions. If $\sigma \in \Sigma^T$ is an equilibrium of the game (μ, v) and if v is bounded by B, then there exists a $5\delta B$-equilibrium σ' of the game (μ', v), with $\sigma'_t = \sigma_t$ for all $t \in C^{1-\delta}_{\mu'}(T^\delta_{\mu,\mu'})$.*

Proof. For ease of notation, define $Q := C^{1-\delta}_{\mu'}(T^\delta_{\mu,\mu'})$, so that Q^t is the t-fold cartesian product of $C^{1-\delta}_{\mu'}(T^\delta_{\mu,\mu'})$ for $t \in T$. For each $t \in Q$, set $\sigma'_t = \sigma_t$. For $t \notin Q$ such that $q_{\mu'}(t) > 0$, let σ'_t be such that $(\sigma'_t)_{t\in T}$ is an equilibrium of the reduced game where each player with a type $t \in Q$ is required to play $\sigma'_t = \sigma_t$. Such an equilibrium exists by Proposition 4.3.6. By construction, σ'_t is a best response to σ' for $t \notin Q$. Hence, it remains to show that σ'_t is a $5\delta B$-best response for a type $t \in Q$. Hence, let $t \in Q$ such that $q_\mu(t) > 0$ and $q_{\mu'}(t) > 0$. By Lemma 4.4.1,

$$q_{\mu'}\left(Q^t \mid t\right) \geq 1 - \delta. \tag{4.10}$$

Furthermore, by the definition of $Q = C^{1-\delta}_{\mu'}(T^\delta_{\mu,\mu'})$, for each $F \in \mathscr{F}_K$,

$$\left|q_\mu(F \mid t) - q_{\mu'}(F \mid t)\right| \leq \delta. \tag{4.11}$$

Let $a \in A$ such that $\sigma_t(a) > 0$, and let $b \in A$. Then,

$$|\varphi_t(a, \sigma'; \mu') - \varphi_t(b, \sigma'; \mu')| \leq \sum_{\theta \in \Omega_K^t \setminus Q^t} q_{\mu'}(\theta \mid t) \left| v_t(a, \sigma'_{(\theta)}) - v_t(b, \sigma'_{(\theta)}) \right| + \sum_{\theta \in Q^t} q_{\mu'}(\theta \mid t) \left| v_t(a, \sigma'_{(\theta)}) - v_t(b, \sigma'_{(\theta)}) \right|. \quad (4.12)$$

The first sum in (4.12) can be evaluated directly. Using (4.10) and that v is bounded by B,

$$\sum_{\theta \in \Omega_K^t \setminus Q^t} q_{\mu'}(\theta \mid t) \left| v_t(a, \sigma'_{(\theta)}) - v_t(b, \sigma'_{(\theta)}) \right| < \delta B. \quad (4.13)$$

To evaluate the second sum in (4.12), first note that for $\theta \in Q^t$, all neighbors play according to σ. As σ is an equilibrium of (μ, v),

$$\sum_{\theta \in Q^t} q_\mu(\theta \mid t) \left| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \right| \leq \sum_{\theta \in \Omega_K^t \setminus Q^t} q_\mu(\theta \mid t) \left| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \right| \quad (4.14)$$

Also, by (4.10) and (4.11), we have that

$$q_\mu\Big(\Omega_K^t \setminus Q^t \;\Big|\; t\Big) \leq 2\delta. \quad (4.15)$$

Combining (4.14) and (4.15), we obtain

$$\sum_{\theta \in Q^t} q_\mu(\theta \mid t) \left| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \right| \leq 2\delta B. \quad (4.16)$$

Let $P^t := \{\theta \in Q^t \mid q_{\mu'}(\theta \mid t) - q_\mu(\theta \mid t) \geq 0\}$ be the set of neighbor type profiles θ in Q^t such that the conditional probability of θ under μ' is at least as high as under μ. Then, by (4.11),

$$\sum_{\theta \in Q^t} \left| \Big(q_{\mu'}(\theta \mid t) - q_\mu(\theta \mid t)\Big) \Big(v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)})\Big) \right| = \sum_{\theta \in P^t} \Big(q_{\mu'}(\theta \mid t) - q_\mu(\theta \mid t)\Big) \left| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \right| + \sum_{\theta \in Q^t \setminus P^t} \Big(q_\mu(\theta \mid t) - q_{\mu'}(\theta \mid t)\Big) \left| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \right| \leq 2\delta B. \quad (4.17)$$

Combining (4.16) and (4.17), we obtain

$$\sum_{\theta \in Q^t} q_{\mu'}(\theta \mid t) \left| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \right| \leq \sum_{\theta \in Q^t} q_\mu(\theta \mid t) \left| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \right| + \sum_{\theta \in Q^t} \left| q_{\mu'}(\theta \mid t) - q_\mu(\theta \mid t) \right| \left| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \right| \leq 4\delta B. \quad (4.18)$$

Combining (4.12), (4.13) and (4.18) gives

$$|\varphi_t(a, \sigma'; \mu') - \varphi_t(b, \sigma'; \mu')| \leq 5\delta B. \qquad \square$$

Proposition 4.5.6 establishes the sufficiency of the condition in Theorem 4.5.3.

Proposition 4.5.6. *Let $\mu, \mu' \in \mathcal{M}$, and let $\delta \in [0, 1]$. Let v be a profile of payoff functions. Suppose that $d^*(\mu, \mu') \leq \delta$. Then, if σ is an equilibrium of the game (μ, v) and v is bounded by B, then there exists a $5\delta B$-equilibrium σ' of the game (μ', v) such that*

$$|\Phi(\sigma; \mu) - \Phi(\sigma'; \mu')| \leq \Big(4 - \delta\Big)\delta B.$$

Proof. For ease of notation, define $Q := C_{\mu'}^{1-\delta}(T_{\mu,\mu'}^{\delta})$. As $d^*(\mu, \mu') \leq \delta$,

$$|q_\mu(F) - q_{\mu'}(F)| \leq \delta \tag{4.19}$$

for all $F \in \mathscr{F}_K$, and

$$q_{\mu'}\Big(\Theta(Q)\Big) \geq 1 - \delta. \tag{4.20}$$

Let $\sigma \in \Sigma^T$ be an equilibrium of (μ, v). By Lemma 4.5.5, there exists a $5\delta B$-equilibrium $\sigma' \in \Sigma^T$ of (μ', v) such that $\sigma'_t = \sigma_t$ for all $t \in Q$. Hence, using (4.20) and Lemma 4.5.4 (with $\alpha = p = 1 - \delta$),

$$\begin{aligned}
&\Big|\Phi(\sigma'; \mu') - \Phi(\sigma; \mu')\Big| \qquad (4.21)\\
&\quad \leq \sum_{\substack{t \in Q:\\ q_{\mu'}(T)>0}} q_{\mu'}(t) \sum_{\theta \in Q^t} q_{\mu'}(\theta \mid t) \sum_{a \in A} |\sigma'_t(a)\, v_t(a, \sigma'_{(\theta)}) - \sigma_t(a)\, v_t(a, \sigma'_{(\theta)})| + \\
&\qquad \sum_{\substack{t \in Q:\\ q_{\mu'}(T)>0}} q_{\mu'}(t) \sum_{\theta \in \Omega_K^t \setminus Q^t} q_{\mu'}(\theta \mid t) \sum_{a \in A} |\sigma'_t(a)\, v_t(a, \sigma'_{(\theta)}) - \sigma_t(a)\, v_t(a, \sigma'_{(\theta)})| + \\
&\qquad \sum_{\substack{t \in T \setminus Q:\\ q_{\mu'}(T)>0}} q_{\mu'}(t) \sum_{\theta \in \Omega_K^t} q_{\mu'}(\theta \mid t) \sum_{a \in A} |\sigma'_t(a)\, v_t(a, \sigma'_{(\theta)}) - \sigma_t(a)\, v_t(a, \sigma'_{(\theta)})| \\
&\quad < 0 + \Big(1 - (1 - \delta)^2\Big) B \\
&\quad = \Big(2 - \delta\Big)\delta B. \qquad (4.22)
\end{aligned}$$

Define the function $\zeta : \Omega_K \to T$ by $\zeta(\theta) = t$ whenever $\theta \in \Omega_K^t$. That is, the function ζ gives the type of a player for each possible neighbor type profile he may have.

Let $P := \{\theta \in \Omega_K \mid q_{\mu'}(\zeta(\theta)) - q_\mu(\zeta(\theta)) \geq 0\}$. Then,

$$\begin{aligned}\left|\Phi(\sigma;\mu') - \Phi(\sigma;\mu)\right| &\leq \sum_{\theta\in P}\Big(q_{\mu'}(\zeta(\theta)) - q_\mu(\zeta(\theta))\Big)\sum_{a\in A}\sigma_{\zeta(\theta)}|v_{\zeta(\theta)}(a,\sigma_{(\theta)})| + \\ &\qquad \sum_{\theta\in\Omega_K\setminus P}\Big(q_\mu(\zeta(\theta)) - q_{\mu'}(\zeta(\theta))\Big)\sum_{a\in A}\sigma_{\zeta(\theta)}|v_{\zeta(\theta)}(a,\sigma_{(\theta)})| \\ &\leq 2\delta B. \end{aligned} \tag{4.23}$$

Combining (4.22) and (4.23) gives the desired result. □

We now establish necessity. Proposition 4.5.7 establishes that $d_0(\mu,\mu')$ should be small for strategic outcomes to be similar (in the sense defined above).

Proposition 4.5.7. *Let $\delta \in [0,1]$, and let $\mu,\mu' \in \mathcal{M}$. If*

$$d_0(\mu,\mu') > \delta,$$

then there exists a profile v of payoff functions with bound $B = 1$ and an equilibrium σ of the game (μ, v) such that for any δ-equilibrium σ' of (μ', v), it holds that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta.$$

Proof. If $d_0(\mu,\mu') > \delta$, there exists a set of neighbor type profiles $F \in \mathscr{F}_K$ such that $|q_\mu(F) - q_{\mu'}(F)| > \delta$. For each $t \in T, t > 0, a \in A, a^{(t)} \in A^t$ and $\theta \in \Omega_K^t$, let

$$v_t(a, a^{(t)}, \theta) = \begin{cases} 1 & \text{if } \theta \in F, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta$$

for *any* two strategy functions $\sigma, \sigma' \in \Sigma^T$. □

Proposition 4.5.8 establishes that strategic outcomes can be very different if $d_1(\mu,\mu')$ is large.

Proposition 4.5.8. *Let $\delta \in [0,1]$, and let $\mu,\mu' \in \mathcal{M}$. If*

$$d_1(\mu,\mu') > \delta,$$

then there exists a profile v of payoff functions with bound $B = 3$ and an equilibrium σ of the game (μ, v) such that for any δ-equilibrium σ' of the game $(\mu' v)$, it holds that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta^2.$$

Proof. As $d_1(\mu, \mu') > \delta$, we have

$$q_\mu\Big(\Theta(C_\mu^{1-\delta}(T^\delta_{\mu,\mu'}))\Big) > 1 - \delta$$

or

$$q_{\mu'}\Big(\Theta(C_{\mu'}^{1-\delta}(T^\delta_{\mu,\mu'}))\Big) > 1 - \delta. \tag{4.24}$$

Without loss of generality, assume that (4.24) holds. Recall that for each $t \notin T^\delta_{\mu,\mu'}$, there exists a set of neighbor type profiles $F_t \in \mathscr{F}_K$ such that

$$q_{\mu'}(F_t \mid t) - q_\mu(F_t \mid t) > \delta.$$

Write $A = \{b^1, b^2, \dots, b^m\}$, where $m \in \mathbb{N}$, and let payoffs be defined as follows.[9] For each $t \in T$, $a^{(t)} \in A^t$ and $\theta \in \Omega^t_K$, let

$$v_t(b^1, a^{(t)}, \theta) := 0,$$

$$v_t(b^2, a^{(t)}, \theta) := \begin{cases} 2 & \text{if } t \in T^\delta_{\mu,\mu'} \text{ and } a^{(t)}_j = b^2 \text{ for some } j \in \{1, \dots, t\}, \\ -\delta & \text{if } t \in T^\delta_{\mu,\mu'} \text{ and } a^{(t)}_j = b^1 \text{ for all } j \in \{1, \dots, t\}, \\ 1 - q_\mu(F_t \mid t) & \text{if } t \notin T^\delta_{\mu,\mu'} \text{ and } \theta \in F_t, \\ -q_\mu(F_t \mid t) & \text{if } t \notin T^\delta_{\mu,\mu'} \text{ and } \theta \notin F_t, \end{cases}$$

and for $\ell \in \{3, \dots, m\}$, let

$$v_t(b^\ell, a^{(t)}, \theta) := -2.$$

Hence, action b^1 always gives a payoff of 0, regardless of the actions and types of a player and his neighbors. For players with type $t \in T^\delta_{\mu,\mu'}$, action b^2 is only profitable if there is at least one neighbor who also takes action b^2. By contrast, the payoffs of b^2 to players with type $t \notin T^\delta_{\mu,\mu'}$ only depends on their neighbor type profile θ: action b^2 is profitable only if θ belongs to F_t. All other actions than b^1 and b^2 are strictly dominated.

Consider the network game of incomplete information (μ, v). In this game, there is an equilibrium $\sigma \in \Sigma^T$ in which all types $t \in T$ choose action b^1 with probability 1. For each type t, expected payoffs are 0, so that type-averaged payoffs are 0. Now consider the game (μ', v). By definition, for each type $t \notin T^\delta_{\mu,\mu'}$, $q_{\mu'}(F_t \mid t) - q_\mu(F_t \mid t) > \delta$. The interim expected payoffs of playing b^2 are then

$$\varphi_t(b^2, \sigma; \mu') = q_{\mu'}(F_t \mid t)\Big(1 - q_\mu(F_t \mid t)\Big) - \Big(1 - q_{\mu'}(F_t \mid t)\Big) q_\mu(F_t \mid t) > \delta$$

9 This game is based on the "infection game" of Kajii and Morris (1998).

for any strategy function $\sigma \in \Sigma^T$. Hence, in any δ-equilibrium, players with type $t \notin T^\delta_{\mu,\mu'}$ will play action b^2. Let

$$\hat{T}^\delta_{\mu,\mu'} := \{t \in T^\delta_{\mu,\mu'} \mid q_\mu(t) > 0\}$$

be the set of types in $T^\delta_{\mu,\mu'}$ that have positive probability under μ, and let $t \in \hat{T}^\delta_{\mu,\mu'}$. If $q_\mu((T^\delta_{\mu,\mu'})^t \mid t) < 1-\delta$, then, with conditional probability at least δ, a player with type t has at least one neighbor who plays b^2. Hence, the interim expected payoffs of b^2 to such a type are at least

$$\delta \cdot 2 - (1-\delta)\cdot\delta > \delta,$$

so that in any δ-equilibrium, players with type $t \in \hat{T}^\delta_{\mu,\mu'}$ such that $q_\mu((T^\delta_{\mu,\mu'})^t \mid t) < 1-\delta$ will play b^2. By a similar argument, players with type $t \in \hat{T}^\delta_{\mu,\mu'}$ such that $q_\mu((B^{1-\delta}_{\mu'}(T^\delta_{\mu,\mu'}))^t \mid t) < 1-\delta$ will play b^2 in any δ-equilibrium. This argument can be iterated any finite number of times. Consequently, all players with type $t \in \hat{T}^\delta_{\mu,\mu'}$ such that $q_\mu((C^{1-\delta}_{\mu'}(T^\delta_{\mu,\mu'}))^t \mid t) < 1-\delta$ will play b^2 in any δ-equilibrium.

By (4.24), the probability that a player has a type $t \notin C_{\mu'}(T^\delta_{\mu,\mu'})$ is greater than δ. As by Lemma 4.4.1 the set $C^{1-\delta}_{\mu'}(T^\delta_{\mu,\mu'})$ is $(1-\delta)$-closed, the probability that a player has a type $t \in \hat{T}^\delta_{\mu,\mu'}$ such that $q_\mu((C^{1-\delta}_{\mu'}(T^\delta_{\mu,\mu'}))^t \mid t) < 1-\delta$ is greater than δ. Hence, in any δ-equilibrium $\sigma' \in \Sigma^T$ of (μ', v), type-averaged expected payoffs are greater than δ^2, so that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta^2. \qquad \square$$

We can now prove Theorem 4.5.3.

Proof. (If) Let v be a profile of payoff functions. By Proposition 4.5.6, for v bounded by B, and for $k \in \mathbb{N}$ such that $5Bd^*(\mu,\mu^k) \leq \varepsilon$,

$$\chi^*(\mu,\mu^k;v,\varepsilon) \leq \left(4 - d^*(\mu,\mu^k)\right)d^*(\mu,\mu^k)B.$$

Hence, for all profiles of payoff functions v that are bounded and for all $\varepsilon > 0$, if $d^*(\mu,\mu^k) \to 0$, then $\chi^*(\mu,\mu^k;v,\varepsilon) \to 0$.

(Only if) Let $\mu,\mu' \in \mathcal{M}$. For $\delta \in [0,1)$, if $d_0(\mu,\mu') > \delta$ or $d_1(\mu,\mu') > \delta$, then, by Propositions 4.5.7 and 4.5.8, there exists a profile of payoff functions v bounded by $B = 3$ and an equilibrium $\sigma \in \Sigma^T$ of (μ, v) such that for any δ-equilibrium $\sigma' \in \Sigma^T$ of (μ', v), $|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta^2$. $\square$

Before we discuss the implications of Theorem 4.5.3 in more detail, some remarks are in order.

Remark 4.5.9. In the current setting, all players with the same payoff function independently implement the same strategies, i.e., strategies do not depend on a player's identity. This does not drive our results. We study the continuity of a given equilibrium correspondence; whether the equilibrium is defined in terms of deviations of individual players or of types, is irrelevant for the question we study. Secondly, in Chapter 3, we have shown that a counterpart of Theorem 4.5.3 holds for Bayesian network games (where the player set is fixed and strategies may depend on a player's identity) when one defines strategic convergence in terms of symmetric Bayesian ε-equilibria.[10] As we discuss more extensively in the next section, convergence of priors in terms of the prior probabilities assigned to local events is necessary and sufficient for strategic convergence when the type set is finite, which is the direct analogue of the condition we identified in Chapter 3. $\triangleleft$

Remark 4.5.10. Our definition of strategic closeness requires that type-averaged expected payoffs be close in equilibria under two priors. An alternative notion would require that with high probability, a player and his neighbors follow the same strategies under the two priors (cf. Monderer and Samet, 1996). Indeed, from the proof of Proposition 4.5.6, it follows that for two priors $\mu, \mu' \in \mathcal{M}$, if $d^*(\mu, \mu') \leq \delta$ for some $\delta \in [0, 1]$, and $\sigma \in \Sigma^T$ is an equilibrium of (μ, v) for a profile v bounded by B, then there is a $5\delta B$-equilibrium of (μ', v) such that the prior probability (either under μ or μ') that a player or his neighbors have a type $t \in T$ such that $\sigma'_t \neq \sigma_t$ is at most $\delta(2 - \delta)$. However, this alone does not imply that the two priors μ and μ' give similar outcomes from a player's ex ante perspective: one should also consider the difference in prior probabilities under μ and μ'. This is done in the last step of the proof of Proposition 4.5.6. Hence, the appropriate definition of strategic closeness in the current setting considers differences in type-average expected payoffs. $\triangleleft$

Remark 4.5.11. Strategic convergence requires that players choose approximate best responses given their type. If, alternatively, we would only have required that they choose approximate best responses before learning their type, i.e., if we would have considered some ex ante or type-averaged notion of approximate equilibrium, then convergence in the weak topology on Ω_K (i.e., $d_0(\mu, \mu^k) \to 0$) is necessary and sufficient for strategic convergence, see Theorem 4.B.9 in Appendix 4.B. $\triangleleft$

Remark 4.5.12. We allow for a player's payoff to depend on the types of his

[10] Notice that the set of symmetric equilibria of a game need not coincide with the set of equilibria in which all players of the same type are required to follow the same strategy. Symmetric equilibria need to be robust to deviations of Harsanyi's player-types, while equilibria need only be robust to deviations of types in the latter case (see Definition 4.3.5).

neighbors. Obviously, for the subclass of games in which a player's payoffs do not depend on his neighbors' types, the condition we derive for strategic convergence is still sufficient, though it may not be necessary. We conjecture that the current conditions cannot be weakened substantially for this subclass of games. ◁

Remark 4.5.13. The assumption that a player's payoffs only depend on the actions and types of his direct neighbors is not essential. Under some suitable modifications and some additional technical assumptions, one could obtain similar results for games in which players' payoffs depend on the actions and types of players that are less than k steps away from them in the network, for arbitrary $k \in \mathbb{N}$. ◁

Theorem 4.5.3 states that two priors are close in a strategic sense if and only if they are similar in terms of two conditions. Firstly, the priors need to be similar in terms of the prior probabilities they assign to all local events, i.e., events that concern a player and his neighbors. The second condition states that with high probability, a player has to have a type such that his conditional beliefs are close under the two priors, and that he believes that it is likely, given his type, that the conditional beliefs of his neighbors are close, and that they believe, given their type, that it is likely... that the conditional beliefs of their neighbors are close, for any number of iterations. An alternative formulation for this second condition is that the set of types with close conditional beliefs needs to be sufficiently cohesive.

Formulating this condition in terms of cohesiveness of the set of types with close conditional beliefs has the advantage that it draws out the parallel between our results and the results of Morris (2000). Morris studies the spread of a certain action by myopic best-reply dynamics on a fixed network with a countable number of players starting from a finite group of players X. He shows that an action will *not* spread contagiously if and only if the network of players not belonging to X contains a large group of players Y that is sufficiently cohesive, in the sense that players from Y interact mostly with other players from Y, who in turn interact mostly with other players from Y, and so on.

Rather than a fixed network of *players*, we consider a random network of players, which induces a fixed interaction structure for the players' *types*. Suppose there is a set of types with small prior probability with disparate conditional beliefs under two priors. This is the analogue of the small (finite) group of players in the setting that Morris studies. Since their conditional beliefs are very different, the strategies (distributions over actions) chosen by these types may be very different under the two priors. We want to know under which conditions these types do not "infect" a large (in terms of ex ante probability) set of types. This is the case precisely when there is a group of types with high prior probability which is sufficiently cohesive, just like contagion is prevented in the setting of Morris when there is a large set of players that are sufficiently cohesive.

Interestingly, this relation between our results and the results of Morris (2000) illustrates how we can use the formal relation between network games with *complete* information and incomplete information games identified by Morris (1997, 2000) to study network games with *incomplete* information by considering the fixed network of types and their correlations induced by the random network of players.

4.5.2 Conditional beliefs and strategic convergence

Theorem 4.5.3 shows that it is not sufficient if two priors assign similar (prior) probabilities to all events in the space of neighbor type profiles for them to be strategically close, as in Bayesian network games (see Chapter 3). In addition, it needs to hold that with high probability, a player has a type such that his conditional beliefs are similar under the two priors, and that he thinks it is likely, given his type, the conditional beliefs of his neighbors are close, and that they think it is likely, given their type, ... that the conditional beliefs of their neighbors are similar, for any number of iterations. In the current section we investigate when this latter condition will be binding.

To shed some light on this, we first investigate when this condition plays no role. We adopt the following definition from Kajii and Morris (1998):

Definition 4.5.14. *A prior $\mu \in \mathcal{M}$ is* insensitive to small probability events *if for each sequence $(\mu^k)_{k\in\mathbb{N}}$ in $\mathcal{M}$,*

$$\lim_{k\to\infty} d_0(\mu, \mu^k) = 0 \Rightarrow \lim_{k\to\infty} d^*(\mu, \mu^k) = 0.$$

In words, a prior $\mu \in \mathcal{M}$ is insensitive to small probability events if a necessary and sufficient condition for strategic convergence of any sequence $(\mu^k)_{k\in\mathbb{N}}$ in $\mathcal{M}$ to μ is that $d_0(\mu, \mu^k)$ converges to zero when k goes to ∞. The next proposition establishes that a necessary and sufficient condition for a prior to be insensitive to small probability events is that it can be approximated on a finite subset of T that is sufficiently closed:

Proposition 4.5.15. *A prior $\mu \in \mathcal{M}$ is insensitive to small probability events if and only if for each $\varepsilon > 0$, there exists a finite set of types $S_\varepsilon \subseteq T$ that is $(1-\varepsilon)$-closed under μ such* *that the probability that a player has a type in S_ε is at least* $1 - \varepsilon$*, i.e.,*

$$q_\mu\Big(\Theta(S_\varepsilon)\Big) \geq 1 - \varepsilon.$$

The proof can be found in Appendix 4.B.

It is easy to see that the following conditions are sufficient for a prior μ to be insensitive to small probability events:

Finite support: The set of types that have positive probability under μ is finite, i.e.,

$$|\{t \in T \mid q_\mu(t) > 0\}| < \infty.$$

Independent types: Neighbors' types are independent, i.e., for all $t \in T$, all $\theta = (\theta_1, \ldots, \theta_t) \in \Omega_K^t$,

$$q_\mu(\theta \mid t) = \frac{t!}{\prod_{k \in T} c_k(\theta)!} \prod_{\substack{s \in T: \\ q_\mu(s) > 0}} \left(\frac{s q_\mu(s)}{\sum_{r \in T} r q_\mu(r)} \right)^{c_s(\theta)},$$

where $c_k(\theta)$ is the number of elements in θ that are equal to k.

Perfect correlation across neighbor types: Players only interact with players of their own type, i.e., for all $t \in T$ such that $q_\mu(t) > 0$, $q_\mu((t, \ldots, t) \mid t) = 1$, where $(t, \ldots, t)$ is a vector in T of length t.

One case of interest in which a prior has finite support is when the number of players is fixed, as in the class of Bayesian network games discussed in the previous chapter. An example of a network belief system with an unbounded number of players and independent types is given in Example 4.2.1. Finally, network belief systems in which types are perfectly correlated are studied by e.g. Ellison (1993).

Proposition 4.5.15 also gives some insight into the question under which conditions a prior is *most* sensitive to small probability events. Consider two priors $\mu, \mu' \in \mathcal{M}$, and let $\delta \in [0, 1]$. Suppose that with probability at least $1 - \delta$, a player has a type $t \in T^\delta_{\mu,\mu'}$, i.e., a type such that his conditional beliefs under μ and μ' are within δ. Let $\Theta_0 \subseteq T^\delta_{\mu,\mu'}$ be the (possibly empty) set of types in $T^\delta_{\mu,\mu'}$ that with high conditional probability interact with types that do *not* belong to $T^\delta_{\mu,\mu'}$, and, for $\ell = 1, 2, \ldots$, let $\Theta_\ell \subseteq (T^\delta_{\mu,\mu'} \setminus \Theta_{\ell-1})$ be the set of types in $T^\delta_{\mu,\mu'} \setminus \Theta_{\ell-1}$ that interact with high conditional probability with types that do not belong to $T^\delta_{\mu,\mu'} \setminus \Theta_{\ell-1}$. If a player has a type in one of the sets Θ_ℓ, his own conditional beliefs are close under the two priors, but, with high conditional probability, he interacts

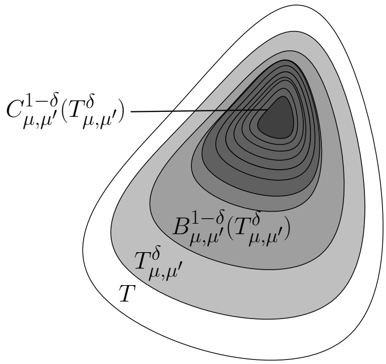


Figure 4.5. Even if with high probability, a player has a type in $T^{\delta}_{\mu,\mu'}$, the probability that he has a type in $C^{1-\delta}_{\mu}(T^{\delta}_{\mu,\mu'})$ may be small.

with types whose conditional beliefs are very different under μ and μ', or who, with high conditionally probability, interact with types whose conditional beliefs are very different under μ and μ', and so on. If the probability is high that a player has such a type, then even if it is a high probability event that a player has a type in $T^{\delta}_{\mu,\mu'}$, the probability that he has a type in $C^{1-\delta}_{\mu}(T^{\delta}_{\mu,\mu'})$ will be small, as illustrated in Figure 4.5. In that case, there will be contagion among types: a player whose conditional beliefs are similar under μ and μ' may be induced to follow a different strategy under μ' than under μ because he thinks it is likely that his neighbors' beliefs are different, or that they think that their neighbors' beliefs are different, and so on. Note that players do not have to interact directly or indirectly to be "infected" by others' behavior. It suffices that players *believe* (given their type) that it is likely that their neighbors believe that . . . their neighbors have a certain type.

Such contagion is ruled out under the following two conditions. Either we need that player's prior is insensitive to small probability events (e.g., the set of types that has positive probability is finite, or types are independent), or we need that the set $T^{\delta}_{\mu,\mu'}$ is sufficiently cohesive, in the sense that all types in $T^{\delta}_{\mu,\mu'}$ interact (with high conditional probability) only with types in $T^{\delta}_{\mu,\mu'}$, who in turn interact only with types in $T^{\delta}_{\mu,\mu'}$, and so on. In that case, if it is a high probability event that a player has a type in $T^{\delta}_{\mu,\mu'}$, it will be a high probability event that a player has a type in $C^{1-\delta}_{\mu}(T^{\delta}_{\mu,\mu'})$. Hence, when there is some correlation among types, but the set $T^{\delta}_{\mu,\mu'}$ is not sufficiently cohesive, players' conditional beliefs play an important role so that small probability events can have a large effect on outcomes.

One implication of this is that one should be careful in defining the game. In particular, it is often assumed in the literature on network games that the size of the network is fixed and that types are independent. The current analysis shows that these assumptions are not innocuous. If players believe that there is some correlation among types and there is uncertainty about the size of the network, then priors may be sensitive to small probability events, which is not the case when the number of players is fixed or when types are independent.

4.6 Conclusions

Given the complexity of many social and economic networks, it is natural to assume that the agents in the network do not have complete information on the structure of the network. We consider a setting in which agents interact strategically on a network and have incomplete information on the network structure. An important question is how sensitive game-theoretic predictions are to the specification of agents' beliefs and information on the network. In the current chapter, we have studied the sensitivity of game-theoretic predictions to assumptions on players' (common) prior in network games of incomplete information. We have asked under what conditions on two priors it holds that for any network game of incomplete information with bounded payoffs in which players hold one of these priors, for any equilibrium in that game, there is an approximate equilibrium in the game with the other prior such that ex ante expected payoffs are close.

Our main result (Theorem 4.5.3) shows that two priors are close in a strategic sense if and only if (*i*) they assign similar prior probabilities to all events involving a player and his neighbors, (*ii*) with high probability, a player believes, given his type, that his neighbors' conditional beliefs are similar under the two priors, and that his neighbors believe, given their type, that... the conditional beliefs of their neighbors are similar, for any number of iterations. Interestingly, this latter condition can also be formulated in terms of correlations among types: an alternative formulation of condition (*ii*) is that the set of types for which conditional beliefs are similar has high probability, and is sufficiently cohesive in the sense that with high conditional probability, a type in that set interacts only with types in that set that, with high conditional probability, only interact with types in that set, and so on.

An important motivation for this work comes from the realization that networks are often large and complex. This suggests that is natural to assume that players on a network have incomplete information about its structure, thus motivating the study of the robustness of game-theoretic predictions to the specification of players' beliefs on their network. There seems to be some tension between this motivation and our results. On the one hand, we assume that players are subject to information constraints. Yet, our results are derived in a setting where players use sophisticated arguments to form expectations over their opponents' behavior. However, it can be shown that the same results are obtained (in the limit) when the game is played repeatedly and to the behavior of his neighbors in the last period (cf. Jackson and Yariv, 2007; Morris, 2000).

To establish our results, we have used ideas and concepts from the literature on higher order beliefs. There are other important questions in the setting of network games of incomplete information that can be answered using ideas from this literature. One important question is how sensitive game-theoretic predictions are to the assumptions on players' information about the network structure. As in much of the literature on network games, we have assumed that players only know their degree. Indeed, Friedkin (1983) finds that the "observational horizon" of individuals is limited in communication networks in organizations: individuals only know their local environment in the network. However, there may well be a large variation among individuals. In addition, players can also represent entities like firms or countries, whose horizon is likely to be larger. For these reasons, it is important to investigate the sensitivity of predictions to informational assumptions. Galeotti et al. (2006) study the effect of varying players' information about the network in a specific setting. Their results indicate that informational assumptions can have an important effect on results. However, to date, there is no systematic investigation how assumptions players' information affects results. The link with the literature on higher order beliefs may also be helpful here, as this literature contains numerous results on the effect of perturbing information structures. The current results suggest that such sensitivity questions are important to study in network games of incomplete information, and they illustrate how one can utilize ideas from the literature on higher order beliefs to answer such questions.

4.A Belief operators for network games

In this section, we extend the definition of the p-belief operator of Monderer and Samet (1989) for Bayesian games to the class of network games of incomplete information. We then discuss its applicability for the problems we study.

First we need some more notation. The p-belief operator of Monderer and Samet (1989) quantifies players' beliefs on "global" events. In the current context, these events would be sets of networks. Since players cannot distinguish between networks that belong to the same isomorphism class, it is possible to speak of a player's belief that the network he is in belongs to a given isomorphism class, or to some collection of isomorphism classes, but we cannot speak of a player's belief that the network he belongs to is some particular network. Technically, a singleton $\{g\}$, where $g \in \mathcal{G}^{(n)}$, or, more generally, a set of networks $\{g_1, \dots, g_\ell\}$, where $g_1, \dots, g_\ell \in \mathcal{G}$, that does not coincide with the union of one or more isomorphism classes cannot be a measurable set in our framework. Hence, the relevant σ-algebra is the σ-algebra generated by the set of singletons of the collection of isomorphism classes; the events we are interested in are sets of networks that consist of one or more isomorphism classes.

This leads us to the following definitions. Let $\mathscr{F}_{\mathscr{C}}$ be the σ-field generated by the set of singletons of the collection of isomorphism classes $\mathscr{C}$. For $G \in \mathscr{F}_{\mathscr{C}}$, let

$$\mathscr{C}_G := \{C \in \mathscr{C} \mid C \cap G \neq \emptyset\}$$

be the set of isomorphism classes of $\mathcal{G}$ contained in G, and let $\mu \in \mathcal{M}$. We define a probability measure r_μ induced by μ on the measurable space $(\mathcal{G}, \mathscr{F}_{\mathscr{C}})$ by

$$\begin{aligned} \forall G \in \mathscr{F}_{\mathscr{C}} : \quad r_\mu(G) &:= \frac{\sum_{C \in \mathscr{C}_G} \mu(C)\, n_C(\Omega_K)}{\sum_{C \in \mathscr{C}} \mu(C)\, n_C(\Omega_K)} \\ &= \frac{1}{\hat{n}} \sum_{C \in \mathscr{C}_G} \mu(C) n_C(\Omega_K). \end{aligned}$$

That is, $r_\mu(G)$ is the prior probability that a player belongs to a network in the set G (cf. Example 4.3.2). For $t \in T$ such that $q_\mu(t) > 0$, let

$$r_\mu(G \mid t) := \frac{\sum_{C \in \mathscr{C}_G} \mu(C)\, n_C(\Omega_K^t)}{\sum_{C \in \mathscr{C}} \mu(C)\, n_C(\Omega_K^t)}$$

be the conditional probability that a player belongs to a network in G given that his type is t. The *p-belief operator for network games of incomplete information* is then defined by

$$\tilde{B}_\mu^p(G) := \{g \in \mathcal{G} \mid \forall i \in V(g), q_\mu(D_i(g)) > 0 \Rightarrow r_\mu(G \mid D_i(g)) \geq p\}.$$

That is, the p-belief operator for network games of incomplete information associates with each set of networks $G \in \mathscr{F}_{\mathscr{C}}$ the set of networks in which all players believe that their network belongs to G with probability at least p, given their type (whenever their type has positive probability). Notice that since for any $G \in \mathscr{F}_{\mathscr{C}}$, for any $g \in \mathcal{G}$, it holds that $g \in \tilde{B}^p_\mu(G)$ if and only $g' \in \tilde{B}^p_\mu(G)$ for all $g' \in \mathcal{G}$ that belong to the same isomorphism class as g, the set $\tilde{B}^p_\mu(G)$ is an element of $\mathscr{F}_{\mathscr{C}}$. Hence, the p-belief operator is a mapping from $\mathscr{F}_{\mathscr{C}}$ to $\mathscr{F}_{\mathscr{C}}$, and can be iterated to define the event (set of networks) that all players believe that their network belongs to $G \in \mathscr{F}_{\mathscr{C}}$ with probability at least p and that all players believe with conditional probability at least p that all players believe that their network belongs to G with conditional probability at least p, and so on.

Unfortunately, the p-belief operator cannot deal with the events we are interested in. We would like to consider sets of networks such as

$$E = \left\{ g \in \mathcal{G} \mid i \in V(g) \Rightarrow D_j(g) \in S \text{ for all } j \in N_i(g) \right\},$$

where $S \subseteq T$. That is, E is the set of networks in which all players have their types in some set S. If $E \in \mathscr{F}_{\mathscr{C}}$, we could use the p-belief operator for network games of incomplete information to obtain the event that all players believe with conditional belief at least p that their neighbors have their type in S, etcetera.

However, the set E is not an event in this setting, i.e., $E \notin \mathscr{F}_{\mathscr{C}}$. The problem is that, while we can only define players' beliefs over isomorphism classes, to use the p-belief operator $\tilde{B}^p_\mu$ for our purposes, we need to consider players' beliefs over the types of particular players. The local p-belief operator has the advantage that this is not necessary: we can consider the probability that an *arbitrary* player has a certain neighbor type profile. On the other hand, the local p-belief operator is ill suited to analyze players' beliefs over global properties of the network, such as the size or component structure of a network. Hence, the p-belief operator of Monderer and Samet (1989) (extended to the class of network games of incomplete information) and the local p-belief operator developed in the current chapter are complementary in the analysis of players' beliefs in network games.

Remark 4.A.1. A possible solution to the problem alluded to above could be to make the additional assumption on priors of *conditional exchangeability*: given that the total number of players is $n \in \mathbb{N}$, the neighbor degree profiles $K_1, \ldots, K_n$ of the n players are exchangeable. We could then exploit this in a way analogous to the way we used exchangeability in the case of Bayesian network games (Chapter 3) to define beliefs over the event that an arbitrary player has a neighbor type profile in some set of neighbor type profiles using the p-belief operator for network games of incomplete information. There is a complication, though, that does not arise in the current approach. When calculating the probability of an event in this setting, one needs to take into account a player's belief that the total number of players is n, for each $n \in \mathbb{N}$, which is not necessary in the current approach. The problem is that by learning his type, a player obtains information about the total number of players. If a player has type $t \in T$, he knows that there are at least $t + 1$ players. More generally, a player of type t will attach a higher weight to large networks than a player of type $t' < t$. It is not immediate how to deal with these issues. $\triangleleft$

4.B Proofs

4.B.1 Proof of Proposition 4.3.6

Proposition 4.3.6 uses Lemma 4.B.1.

Lemma 4.B.1. *Let (μ, v) be a network game of incomplete information such that the profile v of payoff functions is bounded. For each $t \in T$, let the function $\varphi_t(\cdot;\mu)$ on Σ^T be defined as in* (4.2). *Then, $\varphi_t(\cdot;\mu)$ is continuous on the (topological) product space Σ^T.*

Proof. For each $t \in T$ and $n \in \mathbb{N}$, let

$$\Omega_K^{t,n} := \{(k_1, \ldots, k_t) \in \{1, \ldots, n\}^t \mid k_1 \geq k_2 \geq \ldots \geq k_{t-1} \geq k_t\}$$

be the set of neighbor type profiles of a player of type t such that the type of each neighbor is at most n. Clearly, $\Omega_K^{t,n}$ is a finite subset of the countable set Ω_K^t. For each $t \in T$ and $\sigma \in \Sigma^T$, define

$$\varphi_t^{(n)}(\sigma;\mu) := \begin{cases} \sum_{a \in A} \sigma_t(a) \sum_{\theta \in \Omega_K^{t,n}} q_\mu(\theta \mid t)\, v_t\big(a, \sigma_{(\theta)}, \theta\big) & \text{if } q_\mu(t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $t \in T$ such that $q_\mu(t) = 0$, it holds that $\varphi_t^{(n)}(\sigma;\mu) = \varphi_t(\sigma;\mu) = 0$ for all $\sigma \in \Sigma^T$. Let $t \in T$ such that $q_\mu(t) > 0$. By the triangle inequality, for each $\sigma \in \Sigma^T$,

$$\Big|\varphi_t\big(\sigma;\mu\big) - \varphi_t^{(n)}\big(\sigma;\mu\big)\Big| \leq \sum_{\theta \in \Omega_K^t \setminus \Omega_K^{t,n}} q_\mu(\theta \mid t)\, \Big|v_t\big(a, \sigma_{(\theta)}, \theta\big)\Big|.$$

As v is bounded, there exists $B \geq 0$ such that

$$\sum_{\theta \in \Omega_K^t \setminus \Omega_K^{t,n}} q_\mu(\theta \mid t)\, \Big|v_t\big(a, \sigma_{(\theta)}, \theta\big)\Big| \leq B \sum_{\theta \in \Omega_K^t \setminus \Omega_K^{t,n}} q_\mu(\theta \mid t)$$

for all $\sigma \in \Sigma^T$. Moreover,

$$\lim_{n\to\infty} \sum_{\theta\in\Omega_K^t\setminus\Omega_K^{t,n}} q_\mu(\theta \mid t) = 0.$$

Hence, for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $\sigma \in \Sigma^T$,

$$\left|\varphi_t\big(\sigma;\mu\big) - \varphi_t^{(n)}\big(\sigma;\mu\big)\right| \leq \varepsilon \tag{4.25}$$

for all $n > N_\varepsilon$. That is, for each $t \in T$, the sequence $\left(\varphi_t^{(n)}(\cdot;\mu)\right)_{n\in\mathbb{N}}$ converges uniformly on Σ^T to $\varphi_t(\cdot;\mu)$. As for each $n \in \mathbb{N}$, the function $\varphi_t^{(n)}(\cdot;\mu)$ is continuous on Σ^T, the function $\varphi_t(\cdot;\mu)$ is continuous on Σ^T. □

We are now ready to prove Proposition 4.3.6. Consider a network game of incomplete information (μ, v) such that v is bounded, and fix some strategy function $\tau \in \Sigma^T$. Let $n \in \mathbb{N}$, and let $T^{(n)} := \{1, \ldots, n\}$. Recall the definition of the function $\varphi_t(\cdot;\mu)$ on Σ^T in (4.2).

Consider the strategic game $G^{(n)} = \langle T^{(n)}, (\Sigma_t)_{t\in T^{(n)}}, \left(\tilde{\varphi}_t^{(n)}(\cdot;\mu)\right)_{t\in T^{(n)}}\rangle$, where for each $t \in T^{(n)}$, $\Sigma_t = \Sigma$ and $\tilde{\varphi}_t^{(n)}(\cdot;\mu)$ is the real-valued function on Σ^n defined by

$$\forall\sigma^{(n)} \in \Sigma^n : \quad \tilde{\varphi}_t^{(n)}\big(\sigma^{(n)};\mu\big) = \varphi_t\big(\sigma_1^{(n)}, \ldots, \sigma_n^{(n)}, \tau_{n+1}, \tau_{n+2}, \ldots;\mu\big).$$

That is, the payoff of a player $t \in T^{(n)}$ in the game $G^{(n)}$ is the expected payoff of a player of type t in the original game (μ, v), given that players with type $t \in T \setminus T^{(n)}$ play according to τ. The set Σ is a nonempty, convex, compact subset of a finite-dimensional Euclidean space, and for each $t \in T^{(n)}$, $\tilde{\varphi}_t^{(n)}(\cdot;\mu)$ is a continuous real-valued function on Σ^n that is quasi-concave in σ_t on Σ. Hence, the best-response correspondence $b_t : \Sigma^n \rightrightarrows \Sigma^n$ of each player $t \in T^{(n)}$ is nonempty, convex-valued, and upper-hemicontinuous, so that by Kakutani's fixed point theorem, a Nash equilibrium $\left(\bar{\sigma}_1^{(n)}, \ldots, \bar{\sigma}_n^{(n)}\right) \in \Sigma^n$ exists for $G^{(n)}$.

For each $n \in \mathbb{N}$, define

$$\bar{\sigma}^{(n)} := \left(\bar{\sigma}_1^{(n)}, \ldots, \bar{\sigma}_n^{(n)}, \tau_{n+1}, \tau_{n+2}, \ldots\right).$$

The set Σ is compact; hence, by the Cantor diagonal method (e.g. Ok, 2007, pp. 197–198), there exists a subsequence $(\bar{\sigma}^{(n_j)})_{j\in\mathbb{N}}$ of the sequence $(\bar{\sigma}^{(n)})_{n\in\mathbb{N}}$ that converges to some $\bar{\sigma} = (\bar{\sigma}_1, \bar{\sigma}_2, \ldots) \in \Sigma^T$. We claim that $\bar{\sigma}$ is an equilibrium of the original game (μ, v). Suppose not. Then there exists $t \in T$ and $\sigma_t \in \Sigma$ such that

$$\varphi_t(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{t-1}, \bar{\sigma}_t, \bar{\sigma}_{t+1}, \ldots;\mu) < \varphi_t(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{t-1}, \sigma_t, \bar{\sigma}_{t+1}, \ldots;\mu).$$

By Lemma 4.B.1, φ_t is continuous on the topological product space Σ^T. Hence, there exists $j \in \mathbb{N}$ such that $n_j \geq t$ and

$$\varphi_t\Big(\bar{\sigma}_1^{(n_j)}, \ldots, \bar{\sigma}_t^{(n_j)}, \ldots, \bar{\sigma}_{n_j}^{(n_j)}, \tau_{n_j+1}, \tau_{n_j+2}, \ldots; \mu\Big) < \varphi_t\Big(\bar{\sigma}_1^{(n_j)}, \ldots, \sigma_t, \ldots, \bar{\sigma}_{n_j}^{(n_j)}, \tau_{n_j+1}, \tau_{n_j+2}, \ldots; \mu\Big).$$

But this contradicts that $\Big(\bar{\sigma}_1^{(n_j)}, \ldots, \bar{\sigma}_{n_j}^{(n_j)}\Big)$ is a Nash equilibrium of the game $G^{(n_j)}$. □

4.B.2 Properties of the local belief operator

In this section, we prove the properties of the local p-belief operator as listed in Section 4.4, and we prove Lemma 4.4.1 and 4.4.2.

Lemma 4.B.2 (Continuity). *Let $S \subseteq T$, and for $k \in \mathbb{N}$, let $T_k \subseteq T$. If $T_k \downarrow S$, i.e., if $(T_k)_{k\in\mathbb{N}}$ is a decreasing sequence and $\bigcap_{k\in\mathbb{N}} T_k = S$, then $B_\mu^p(T_k) \downarrow B_\mu^p(S)$.*

Proof. First note that $B_\mu^p(T_{k+1}) \subseteq B_\mu^p(T_k)$ for all $k \in \mathbb{N}$, i.e., $(B_\mu^p(T_k))_{k\in\mathbb{N}}$ is a decreasing sequence. It remains to show that

$$\bigcap_{k\in\mathbb{N}} B_\mu^p(T_k) = B_\mu^p\Big(\bigcap_{k\in\mathbb{N}} T_k\Big).$$

First suppose $t \in \bigcap_{k\in\mathbb{N}} B_\mu^p(T_k)$. Then, obviously, $t \in T_k$ for all $k \in \mathbb{N}$. We need to distinguish two cases: $q_\mu(t) = 0$ and $q_\mu(t) > 0$. Suppose that $q_\mu(t) = 0$. Then, by definition, $t \in B_\mu^p(\bigcap_{k\in\mathbb{N}} T_k)$. So suppose $q_\mu(t) > 0$. Then, $q_\mu(T_k^t \mid t) \geq p$ for all $k \in \mathbb{N}$. Furthermore, $(T_k^t)_{k\in\mathbb{N}}$ is a decreasing sequence, and $\bigcap_{k\in\mathbb{N}} T_k^t = S^t$. Hence (e.g. Grimmett and Stirzaker, 1992, Lemma 1.3.5),

$$\lim_{k\to\infty} q_\mu\Big(T_k^t \mid t\Big) = q_\mu\Big(\bigcap_{k\in\mathbb{N}} T_k^t \,\Big|\, t\Big).$$

Combining these results gives

$$q_\mu\Big(\bigcap_{k\in\mathbb{N}} T_k^t \,\Big|\, t\Big) \geq p,$$

and hence $t \in B_\mu^p\Big(\bigcap_{k\in\mathbb{N}} T_k^t\Big)$.

Secondly, suppose that $t \in B_\mu^p\Big(\bigcap_{k\in\mathbb{N}} T_k\Big)$. Then, obviously, $t \in T_k$ for all $k \in \mathbb{N}$. Again, we need to consider two cases. If $q_\mu(t) = 0$, then it follows directly from the definition of B_μ^p that $t \in B_\mu^p(T_k)$ for all $k \in \mathbb{N}$, and therefore $t \in \bigcap_{k\in\mathbb{N}} B_\mu^p(T_k)$. So suppose $q_\mu(t) > 0$. Then, $q_\mu(S^t \mid t) \geq p$ implies that $q_\mu(T_k \mid t) \geq p$ for all $k \in \mathbb{N}$. Hence, $t \in B_\mu^p(T_k)$ for all $k \in \mathbb{N}$, and $t \in \bigcap_{k\in\mathbb{N}} B_\mu^p(T_k)$. □

Lemma 4.B.3 (Monotonicity). *For any $T', T'' \subseteq T$, if $T' \subseteq T''$, then $B_\mu^p(T') \subseteq B_\mu^p(T'')$.*

Proof. If $T' \subseteq T''$, then $T' \cap T'' = T'$. Hence,

$$B_\mu^p(T') = B_\mu^p(T' \cap T'') = B_\mu^p(T') \cap B_\mu^p(T'') \subseteq B_\mu^p(T''). \qquad \square$$

Lemma 4.B.4 (Continuity in *p*). *If $p_k \uparrow p$, then, for any $S \subseteq T$, $B_\mu^{p_k}(S) \downarrow B_\mu^p(S)$.*

Proof. Let $S \subseteq T$. It follows directly from the definition of the local p-belief operator that $\left(B_\mu^{p_k}(S)\right)_{k\in\mathbb{N}}$ is a decreasing sequence. It remains to show that

$$\bigcap_{k\in\mathbb{N}} B_\mu^{p_k}(S) = B_\mu^p(S).$$

Suppose $t \in \bigcap_{k\in\mathbb{N}} B_\mu^{p_k}(S)$. If $q_\mu(t) = 0$, then it follows directly from the definition that $t \in B_\mu^p(S)$. So suppose $q_\mu(t) > 0$. Then, $q_\mu(S^t \mid t) \geq p_k$ for all $k \in \mathbb{N}$, and therefore $q_\mu(S^t \mid t) \geq p$. Hence, $t \in B_\mu^p(S)$.

Conversely, suppose $t \in B_\mu^p(S)$. If $q_\mu(t) = 0$, then it follows directly that $t \in \bigcap_{k\in\mathbb{N}} B_\mu^{p_k}(S)$. So suppose $q_\mu(t) > 0$. Then, $q_\mu(S^t) \geq p$, and hence $q_\mu(S^t) \geq p_k$ for all $k \in \mathbb{N}$. We conclude that $t \in B_\mu^{p_k}(S)$ for all k, and hence $t \in \bigcap_{k\in\mathbb{N}} B_\mu^{p_k}(S)$. □

Finally, we present the proofs of Lemmas 4.4.1 and 4.4.2.

Proof of Lemma 4.4.1. By definition, $B_\mu^p\big(C_\mu^p(S)\big) \subseteq C_\mu^p(S)$. It remains to show that $B_\mu^p\big(C_\mu^p(S)\big) \supseteq C_\mu^p(S)$. Obviously, $\left([B_\mu^p]^\ell(S)\right)_{\ell\in\mathbb{N}}$ is a weakly decreasing sequence, and, by definition, $\bigcap_{\ell\in\mathbb{N}}\left[B_\mu^p\right]^\ell(S) = C_\mu^p(S)$. Hence, using that the local p-belief operator is continuous,

$$C_\mu^p(S) = \bigcap_{\ell\in\mathbb{N}}\left[B_\mu^p\right]^\ell(S) \subseteq \bigcap_{\ell\in\mathbb{N}:k\geq 2}\left[B_\mu^p\right]^\ell(S) = B_\mu^p\Big(\bigcap_{\ell\in\mathbb{N}}\left[B_\mu^p\right]^\ell(S)\Big) = B_\mu^p\big(C_\mu^p(S)\big). \qquad \square$$

Proof of Lemma 4.4.2. Suppose $t \in C_\mu^p(T')$. By Lemma 4.4.1, the set $C_\mu^p(T')$ is p-closed. Also, by definition, $C_\mu^p(T') \subseteq B_\mu^p(T')$. Hence, we can set $S = C_\mu^p(T')$, and the statement follows.

Conversely, let $S \subseteq T$ be such that $t \in S$, and

$$S \subseteq B^p_\mu(S), \tag{4.26}$$
$$S \subseteq B^p_\mu(T'). \tag{4.27}$$

We show by induction on ℓ that $S \subseteq \left[B^p_\mu\right]^\ell(T')$ for all $\ell \in \mathbb{N}$, from which it follows that $t \in C^p_\mu(T')$. By (4.27), $S \subseteq \left[B^p_\mu\right]^1(T')$. For each $\ell \in \mathbb{N}$, if $S \subseteq \left[B^p_\mu\right]^\ell(T')$, then by (4.26) and by monotonicity of the local p-belief operator,

$$S \subseteq B^p_\mu(S) \subseteq B^p_\mu\Big(\left[B^p_\mu\right]^\ell(T')\Big) = \left[B^p_\mu\right]^{\ell+1}(T'). \qquad \square$$

4.B.3 Proof of Lemma 4.5.4

By Lemma 4.4.1, $C^p_\mu(S)$ is p-closed. Hence, for all $t \in C^p_\mu(S)$ such that $q_\mu(t) > 0$, $q_\mu\Big((C^p_\mu(S))^t \;\Big|\; t\Big) \geq p$. This yields:

$$\begin{aligned}
q_\mu\Big(\bigcup_{t \in C^p_\mu(S)} \left(C^p_\mu(S)\right)^t\Big) &= \sum_{\substack{t' \in C^p_\mu(S):\\ q_\mu(t')>0}} q_\mu\Big(\bigcup_{t \in C^p_\mu(S)} \left(C^p_\mu(S)\right)^t \Big| t'\Big) q_\mu(t') \\
&= \sum_{\substack{t' \in C^p_\mu(S):\\ q_\mu(t')>0}} q_\mu\Big((C^p_\mu(S))^{t'} \;\Big|\; t'\Big) q_\mu(t') \\
&\geq p \sum_{t' \in C^p_\mu(S)} q_\mu(t') \\
&\geq \alpha p.
\end{aligned} \qquad \square$$

Remark 4.B.5. Note that Lemma 4.5.4 can be generalized: we can replace $C^p_\mu(S)$ in the lemma by any subset of T that is p-closed. We have presented it in its current form for expositional reasons. ◁

4.B.4 Continuity of the type-averaged equilibrium correspondence

In an (approximate) equilibrium, as we defined it, players are required to choose best responses *given their type*, i.e., equilibria are defined in terms of expected payoffs. Alternatively, we could define equilibrium in terms of type-averaged expected payoffs, allowing types with low prior probability to follow strategies that are suboptimal. In standard Bayesian games, lower hemicontinuity of the ex ante ε-equilibrium has been studied by Engl (1995). He shows that the weak topology is sufficient for lower hemicontinuity of the ex ante ε-equilibrium in countable state spaces. Here, we derive an analogous result for the type-averaged ε-equilibrium correspondence (see below for a precise definition). We show that the weak topology is sufficient (and also necessary) to guarantee lower-hemicontinuity of this correspondence.

First we need some definitions. Recall the definition of type-averaged expected payoffs from Section 4.3.

Definition 4.B.6. *Let $\varepsilon \geq 0$. A strategy function $\sigma \in \Sigma^T$ is a* type-averaged ε-equilibrium *of a network game of incomplete information (μ, v) if*

$$\Phi(\sigma; \mu) \geq \Phi(\sigma'; \mu) - \varepsilon$$

for all $\sigma' \in \Sigma$. We refer to a type-averaged 0-equilibrium as a type-averaged equilibrium*.*

Proposition 4.B.7. *Let (μ, v) be a network game of incomplete information. If the profile of payoff functions v is bounded, the game has a type-averaged equilibrium.*

The proof follows directly from the proof of Proposition 4.3.6. Let $\mathcal{N}_\tau^\varepsilon(\mu, v)$ denote the set of type-averaged ε-equilibria of (μ, v).

We define a notion of strategic convergence for the current setting. Let $\mu, \mu' \in \mathcal{M}$, and let v be a profile of payoff functions. For $\varepsilon \geq 0$, define

$$\chi_\tau(\mu, \mu'; v, \varepsilon) := \sup_{\sigma \in N_\tau^0(\mu, v)} \inf_{\sigma' \in N_\tau^\varepsilon(\mu', v)} |\Phi(\sigma; \mu) - \Phi(\sigma'; \mu')|,$$

and

$$\chi_\tau^*(\mu, \mu; v, \varepsilon) := \max\{\chi_\tau(\mu, \mu'; v, \varepsilon), \chi_\tau(\mu', \mu; v, \varepsilon)\}.$$

Definition 4.B.8. *Take any $\mu \in \mathcal{M}$, and consider a sequence $(\mu^k)_{k \in \mathbb{N}}$ in $\mathcal{M}$. The sequence $(\mu^k)_{k \in \mathbb{N}}$* converges strategically in the sense of type-averaged expected payoffs *to μ if for each profile v of payoff functions that is bounded, for each $\varepsilon > 0$, we have that*

$$\lim_{k \to \infty} \chi_\tau^*(\mu, \mu^k; v, \varepsilon) = 0.$$

Recall the definition of d_0 from Section 4.5, and notice that the metric d_0 generates the weak topology.

Theorem 4.B.9. *Let $\mu \in \mathcal{M}$ and let $(\mu^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{M}$. Then, $(\mu^k)_{k\in\mathbb{N}}$ converges strategically in the sense of type-averaged expected payoffs to μ if and only if*

$$\lim_{k\to\infty} d_0(\mu, \mu^k) = 0.$$

The proof follows from Proposition 4.B.10 and 4.B.11.

Proposition 4.B.10 establishes that the weak topology is sufficient.

Proposition 4.B.10. *Let $\mu, \mu' \in \mathcal{M}$, and let $\delta \in [0,1]$. Let v be a profile of payoff functions. Suppose that $d_0(\mu, \mu') \leq \delta$. Then, if $\sigma \in \Sigma^T$ is a type-averaged equilibrium of the game (μ, v) and if v is bounded by B, then σ is a type-averaged $4\delta B$-equilibrium of the game (μ', v), and*

$$|\Phi(\sigma; \mu) - \Phi(\sigma; \mu')| \leq 2\delta B.$$

Proof. Let $t \in T$ be such that $q_\mu(t) > 0$, and let $\sigma'_t \in \Sigma$. As σ is a type-averaged equilibrium of (μ, v),

$$\Phi(\sigma'; \mu) - \Phi(\sigma; \mu) \leq 0.$$

Hence,

$$\Phi(\sigma; \mu') - \Phi(\sigma'; \mu') \geq \Phi(\sigma; \mu') - \Phi(\sigma; \mu) + \Phi(\sigma'; \mu) - \Phi(\sigma'; \mu'). \tag{4.28}$$

As $d_0(\mu, \mu') \leq \delta$,

$$\Phi(\sigma; \mu) - \Phi(\sigma; \mu') \geq -2\delta B, \tag{4.29}$$

$$\Phi(\sigma'; \mu) - \Phi(\sigma'; \mu') \geq -2\delta B. \tag{4.30}$$

Substituting (4.29) and (4.30) in (4.28), we find

$$\Phi(\sigma; \mu') \geq \Phi(\sigma'; \mu') - 4\delta B,$$

proving the first claim. The second claim follows directly from (4.29). □

The next proposition shows that the topology generated by d_0 is also necessary.

Proposition 4.B.11. *Let $\delta \in [0,1]$, and let $\mu, \mu' \in \mathcal{M}$. If*

$$d_0(\mu, \mu') > \delta,$$

then there exists a profile v of payoff functions with bound $B = 1$ and an equilibrium σ of the game (μ, v) such that for any δ-equilibrium σ' of (μ', v), it holds that

$$|\Phi(\sigma; \mu) - \Phi(\sigma'; \mu')| > \delta.$$

The proof follows directly from the proof of Proposition 4.5.7.

We can now prove Theorem 4.B.9.

Proof. (If) Let v be a profile of payoff functions. By Proposition 4.B.10, for v bounded by B, and for $k \in \mathbb{N}$ such that $4Bd_0(\mu, \mu^k) \leq \varepsilon$,

$$\chi^*_\tau(\mu, \mu^k; v, \varepsilon) \leq 2d_0(\mu, \mu^k)B.$$

Hence, for all profiles of payoff functions v that are bounded and for all $\varepsilon > 0$, if $d_0(\mu, \mu^k) \to 0$, then $\chi^*_\tau(\mu, \mu^k; v, \varepsilon) \to 0$.

(Only if) Let $\mu, \mu' \in \mathcal{M}$. For $\delta \in [0, 1)$, if $d_0(\mu, \mu') > \delta$, then, by Proposition 4.B.11, there exists a profile of payoff functions v bounded by $B = 1$ and a type-averaged equilibrium $\sigma \in \Sigma^T$ of (μ, v) such that for any type-averaged δ-equilibrium $\sigma' \in \Sigma^T$ of (μ', v), $|\Phi(\sigma; \mu) - \Phi(\sigma'; \mu')| > \delta$. □

4.B.5 Proof of Proposition 4.5.15

Proposition 4.5.15 uses Lemma 4.B.12.

Lemma 4.B.12. *Let $\mu \in \mathcal{M}$, and let $(\mu^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{M}$. If*

$$\lim_{k\to\infty} d_0(\mu, \mu^k) = 0 \qquad \textit{and} \qquad \lim_{k\to\infty} d_1(\mu, \mu^k) = 0,$$

then

$$\lim_{k\to\infty} d_1(\mu^k, \mu) = 0.$$

Proof. Let $\varepsilon > 0$. By assumption, there exists $K \in \mathbb{N}$ such that for all $k > K$,

$$\sup_{F\in\mathscr{F}_K} |q_\mu(F) - q_{\mu^k}(F)| \leq \frac{\varepsilon}{2}, \tag{4.31}$$

and

$$\inf\left\{\delta \in [0, 1] \;\middle|\; q_\mu\left(\Theta(C^{1-\delta}_\mu(T^\delta_{\mu,\mu^k}))\right)\right\} \leq \frac{\varepsilon}{2}. \tag{4.32}$$

Let $k > K$. Recall that for $t \in T^{\varepsilon/2}_{\mu,\mu^k}$ such that $q_\mu(t) > 0$ and $q_{\mu^k}(t) > 0$,

$$\sup_{F\in\mathscr{F}_K} |q_\mu(F \mid t) - q_{\mu^k}(F \mid t)| \leq \frac{\varepsilon}{2}, \tag{4.33}$$

and define

$$\hat{T}^{\varepsilon/2}_{\mu,\mu^k} := \Big\{ t \in T^{\varepsilon/2}_{\mu,\mu^k} \;\Big|\; q_\mu(t) > 0 \Big\}.$$

Note that, unlike $T^{\varepsilon/2}_{\mu,\mu^k}$, the set $\hat{T}^{\varepsilon/2}_{\mu,\mu^k}$ is not symmetric in μ and μ^k, i.e., $\hat{T}^{\varepsilon/2}_{\mu^k,\mu} \neq \hat{T}^{\varepsilon/2}_{\mu,\mu^k}$. Using (4.33) and the fact that the local p-belief operator is monotonic, we obtain

$$B^{1-\varepsilon/2}_{\mu}\Big(\hat{T}^{\varepsilon/2}_{\mu,\mu^k}\Big) \subseteq B^{1-\varepsilon}_{\mu^k}\Big(\hat{T}^{\varepsilon/2}_{\mu,\mu^k}\Big) \subseteq B^{1-\varepsilon}_{\mu^k}\Big(\hat{T}^{\varepsilon}_{\mu,\mu^k}\Big).$$

Hence,

$$C^{1-\varepsilon/2}_{\mu}\Big(\hat{T}^{\varepsilon/2}_{\mu,\mu^k}\Big) \subseteq C^{1-\varepsilon}_{\mu^k}\Big(\hat{T}^{\varepsilon}_{\mu,\mu^k}\Big).$$

Using this and (4.32), we obtain

$$q_\mu\Big(\Theta(C^{1-\varepsilon}_{\mu^k}(T^{\varepsilon}_{\mu,\mu^k}))\Big) \geq q_\mu\Big(\Theta(C^{1-\varepsilon}_{\mu^k}(\hat{T}^{\varepsilon}_{\mu,\mu^k}))\Big) \geq$$
$$q_\mu\Big(\Theta(C^{1-\varepsilon/2}_{\mu}(\hat{T}^{\varepsilon/2}_{\mu,\mu^k}))\Big) = q_\mu\Big(\Theta(C^{1-\varepsilon/2}_{\mu}(T^{\varepsilon/2}_{\mu,\mu^k}))\Big) \geq 1 - \frac{\varepsilon}{2},$$

so that by (4.31),

$$q_{\mu^k}\Big(\Theta(C^{1-\varepsilon}_{\mu^k}(T^{\varepsilon}_{\mu,\mu^k}))\Big) \geq 1 - \varepsilon.$$

Combining these results gives

$$\inf\Big\{\delta \in [0,1] \mid q_{\mu^k}\Big(\Theta(C^{1-\delta}_{\mu^k}(T^{\delta}_{\mu,\mu^k}))\Big) \geq 1 - \delta\Big\} \leq \varepsilon. \qquad \square$$

We can now prove Proposition 4.5.15.

(If) Let $\varepsilon > 0$, and let $(\mu^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{M}$. Suppose that $S_\varepsilon \subseteq T$ is such that

$$|S_\varepsilon| < \infty, \tag{4.34}$$
$$S_\varepsilon \subseteq B^{1-\varepsilon}_{\mu}(S_\varepsilon), \tag{4.35}$$
$$q_\mu\Big(\Theta(S_\varepsilon)\Big) \geq 1 - \varepsilon. \tag{4.36}$$

By Lemma 4.B.12, if $d_0(\mu,\mu^k) \to 0$ and $d_1(\mu,\mu^k) \to 0$, then also $d_1(\mu^k,\mu) \to 0$. Hence, it is sufficient to show that $d_1(\mu,\mu^k) \to 0$ whenever $d_0(\mu,\mu^k) \to 0$.

Let $\hat{S}_\varepsilon := \{t \in S_\varepsilon \mid q_\mu(t) > 0\}$ be the set of types in S_ε that have positive probability under μ. By (4.34), there exists $c > 0$ such that $q_\mu(t) = q_\mu(\Omega^t_K) \geq c$ for all

$t \in \hat{S}_\varepsilon$. Then, for all $k \in \mathbb{N}$, for all $t \in \hat{S}_\varepsilon$,

$$\begin{aligned}\sup_{F\in\mathscr{F}_K} |q_\mu(F \mid t) - q_{\mu^k}(F \mid t)| &= \sup_{F\in\mathscr{F}_K} \left| \frac{q_\mu(F\cup\Omega^t_K)}{q_\mu(\Omega^t_K)} - \frac{q_{\mu^k}(F\cup\Omega^t_K)}{q_\mu(\Omega^t_K)} + \right. \\ &\qquad\qquad \left. \frac{q_{\mu^k}(F\cup\Omega^t_K)}{q_\mu(\Omega^t_K)} - \frac{q_{\mu^k}(F\cup\Omega^t_K)}{q_{\mu^k}(\Omega^t_K)} \right| \\ &\leq \sup_{F\in\mathscr{F}_K} \frac{1}{q_\mu(t)} \left| q_\mu(F\cup\Omega^t_K) - q_{\mu^k}(F\cup\Omega^t_K) \right| + \\ &\qquad\qquad \sup_{F\in\mathscr{F}_K} \frac{q_{\mu^k}(F \mid t)}{q_\mu(t)} \left| q_\mu(\Omega^t_K) - q_{\mu^k}(\Omega^t_K) \right| \\ &\leq \left(\frac{2}{c}\right) \cdot \sup_{F\in\mathscr{F}_K} |q_\mu(F) - q_{\mu^k}(F)|. \end{aligned} \tag{4.37}$$

Suppose that $\lim_{k\to\infty} d_0(\mu, \mu^k) = 0$. Then there exists $K \in \mathbb{N}$ such that for all $k > K$,

$$\sup_{F\in\mathscr{F}_K} |q_\mu(F) - q_{\mu^k}(F)| \leq \left(\frac{c}{2}\right) \cdot \varepsilon.$$

Let $k > K$. Then, by (4.37), for all $t \in \hat{S}_\varepsilon$ such that $q_{\mu^k}(t) > 0$, it holds that

$$\sup_{F\in\mathscr{F}_K} |q_\mu(F \mid t) - q_{\mu^k}(F \mid t)| \leq \varepsilon,$$

so that $S_\varepsilon \subseteq T^\varepsilon_{\mu,\mu^k}$. By monotonicity of the local p-belief operator and (4.35),

$$S_\varepsilon = B^{1-\varepsilon}_\mu(S_\varepsilon) \subseteq B^{1-\varepsilon}_\mu\left(T^\varepsilon_{\mu,\mu^k}\right).$$

Using Lemma 4.4.2 and (4.35), we obtain

$$t \in S_\varepsilon \Rightarrow t \in C^{1-\varepsilon}_\mu\left(T^\varepsilon_{\mu,\mu^k}\right),$$

so that (using (4.36))

$$q_\mu\left(\Theta(C^{1-\varepsilon}_\mu(T^\varepsilon_{\mu,\mu^k}))\right) \geq q_\mu\left(\Theta(S_\varepsilon)\right) \geq 1 - \varepsilon.$$

Hence, $d_1(\mu, \mu^k) \leq \varepsilon$ whenever $d_0(\mu, \mu^k) \leq \left(\frac{c}{2}\right)\varepsilon$.

(Only if) Suppose that

$$\lim_{k\to\infty} d_0(\mu, \mu^k) = 0 \Rightarrow \lim_{k\to\infty} d_1(\mu, \mu^k).$$

First we show that there exists a sequence $(\nu^k)_{k\in\mathbb{N}}$ in $\mathcal{M}$ such that

(a) for each $k \in \mathbb{N}$, the set of types $\{t \in T \mid q_{\nu^k}(t) > 0\}$ that have positive probability under ν^k is finite;

(b) $(\nu^k)_{k \in \mathbb{N}}$ converges to μ in the weak topology on Ω_K:

$$\lim_{k\to\infty} \sup_{F \in \mathscr{F}_K} |q_\mu(F) - q_{\nu^k}(F)| = 0.$$

The sequence $(\nu^k)_{k \in \mathbb{N}}$ is easy to construct. If μ has finite support in T, i.e., if the set $\{t \in T \mid q_\mu(t) > 0\}$ is finite, then simply set $\nu^k = \mu$ for all $k \in \mathbb{N}$. Otherwise, we construct $(\nu^k)_{k \in \mathbb{N}}$ as follows. For each $k \in \mathbb{N}$, define

$$\hat{\mathcal{G}}^{(k)} := \{g \in \mathcal{G} \mid \forall i \in V(g), D_i(g) \leq k\}$$

to be the set of networks in which the maximum degree is k. Note that the sequence $(\hat{\mathcal{G}}^{(k)})_{k \in \mathbb{N}}$ is increasing. For each $g \in \mathcal{G}$, let

$$\nu^k(g) \begin{cases} \frac{\mu(g)}{\mu(\hat{\mathcal{G}}^{(k)})} & \text{if } g \in \hat{\mathcal{G}}^{(k)} \text{ and } \mu(\hat{\mathcal{G}}^{(k)}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that (a) is satisfied. To see that (b) is also satisfied, define $\hat{\mathscr{C}}^{(k)}$ to be the collection of isomorphism classes in $\hat{\mathcal{G}}^{(k)}$ for $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ such that $\mu(\hat{\mathcal{G}}^{(k)}) > 0$, we have

$$\begin{aligned}
\sup_{F \in \mathscr{F}_K} |q_\mu(F) - q_{\nu^k}(F)| &= \sup_{F \in \mathscr{F}_K} \left| \frac{\sum_{C \in \mathscr{C}} \mu(C)\, n_C(F)}{\sum_{C \in \mathscr{C}} \mu(C)\, n_C(\Omega_K)} - \frac{\sum_{C \in \mathscr{C}} \nu^k(C)\, n_C(F)}{\sum_{C \in \mathscr{C}} \nu^k(C)\, n_C(\Omega_K)} \right| \\
&= \sup_{F \in \mathscr{F}_K} \left| \frac{\sum_{C \in \mathscr{C}} \mu(C)\, n_C(F)}{\sum_{C \in \mathscr{C}} \mu(C)\, n_C(\Omega_K)} - \frac{\sum_{C \in \hat{\mathscr{C}}^{(k)}} \nu^k(C)\, n_C(F)}{\sum_{C \in \hat{\mathscr{C}}^{(k)}} \nu^k(C)\, n_C(\Omega_K)} \right| \\
&\leq \frac{1}{\hat{n}} \sup_{F \in \mathscr{F}_K} \left| \sum_{C \in \mathscr{C}} \mu(C)\, n_C(F) - \sum_{C \in \hat{\mathscr{C}}^{(k)}} \mu(C)\, n_C(F) \right| + \\
&\qquad \left(\frac{1}{\hat{n}} - \frac{1}{\sum_{C \in \hat{\mathscr{C}}^{(k)}} \mu(C)\, n_C(\Omega_K)} \right) \sup_{F \in \mathscr{F}_K} \sum_{C \in \hat{\mathscr{C}}^{(k)}} \mu(C)\, n_C(F) \\
&\leq \frac{1}{\hat{n}} \sup_{F \in \mathscr{F}_K} \left(\sum_{C \in \mathscr{C} \setminus \hat{\mathscr{C}}^{(k)}} \mu(C)\, n_C(F) \right) + 1 - \frac{\hat{n}}{\sum_{C \in \hat{\mathscr{C}}^{(k)}} \mu(C)\, n_C(\Omega_K)}.
\end{aligned}$$

As for all $F \in \mathscr{F}_K$,

$$\lim_{k\to\infty} \sum_{C \in \hat{\mathscr{C}}^{(k)}} \mu(C)\, n_C(F) = \sum_{C \in \mathscr{C}} \mu(C)\, n_C(F),$$

it follows that (b) holds.

Since μ is insensitive to small probability events, we also have that $d_1(\mu, \nu^k) \to 0$. Hence, for all $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $k > K$,

$$\sup_{F \in \mathscr{F}_K} |q_\mu(F) - q_{\nu^k}(F)| \leq \frac{\varepsilon}{3} \tag{4.38}$$

and

$$\inf\Big\{\delta \in [0,1] \;\Big|\; q_\mu\Big(\Theta(C_\mu^{1-\delta}(T^\delta_{\mu,\nu^k}))\Big) \geq 1 - \delta\Big\} \leq \frac{\varepsilon}{3}. \tag{4.39}$$

Let $k > K$, and define

$$\hat{T}^\varepsilon_{\mu,\nu^k} := \Big\{t \in T^\varepsilon_{\mu,\nu^k} \;\Big|\; q_{\nu^k}(t) > 0\Big\}$$

to be the set of types in $T^\varepsilon_{\mu,\nu^k}$ that have positive probability under ν^k. By (4.39) and using that the local p-belief operator is monotonic and continuous in p,

$$q_\mu\Big(\Theta(C_\mu^{1-\varepsilon}(T^\varepsilon_{\mu,\nu^k}))\Big) \geq q_\mu\Big(\Theta(C_\mu^{1-\varepsilon/3}(T^\varepsilon_{\mu,\nu^k}))\Big) \geq q_\mu\Big(\Theta(C_\mu^{1-\varepsilon/3}(T^{\varepsilon/3}_{\mu,\nu^k}))\Big) \geq 1 - \frac{\varepsilon}{3},$$

so that by (4.38),

$$q_{\nu^k}\Big(\Theta(C_\mu^{1-\varepsilon}(\hat{T}^\varepsilon_{\mu,\nu^k}))\Big) = q_{\nu^k}\Big(\Theta(C_\mu^{1-\varepsilon}(T^\varepsilon_{\mu,\nu^k}))\Big) \geq 1 - \frac{2\varepsilon}{3},$$

and hence (using (4.38) again),

$$q_\mu\Big(\Theta(C_\mu^{1-\varepsilon}(\hat{T}^\varepsilon_{\mu,\nu^k}))\Big) \geq 1 - \varepsilon.$$

By definition, $\hat{T}^\varepsilon_{\mu,\nu^k}$ and hence $C_\mu^{1-\varepsilon}\Big(\hat{T}^\varepsilon_{\mu,\nu^k}\Big)$ are finite. Moreover, by Lemma 4.4.1, $C_\mu^{1-\varepsilon}\Big(\hat{T}^\varepsilon_{\mu,\nu^k}\Big)$ is $(1-\varepsilon)$-closed. Hence, by setting

$$S_\varepsilon = C_\mu^{1-\varepsilon}\Big(\hat{T}^\varepsilon_{\mu,\nu^k}\Big)$$

we obtain the desired result. □

5 Random networks with a group structure

Summary

The current chapter, which is based on Deijfen and Kets (2007), develops a random network model with a group structure, and characterizes its degree distribution and clustering. The clustering of a random network model is defined as the probability that two agents are connected given that they have a common neighbor. The model proposed in the current chapter is a suitable model for many social and economic settings, and provides a flexible framework to model players' beliefs over the group structure in network games with incomplete information on the network structure.

5.1 Introduction

In many social and economic contexts, agents interact in communities or groups. For instance, individuals belong to a family, work at an office, frequent a sports' club, etcetera. Similarly, firms participate in different R&D alliances with several different partners, and they compete with different firms on different markets. The different R&D alliances or markets can be seen as groups. Also the coauthorship networks that have recently attracted attention in economics (Goyal et al., 2006) can be seen as an example. In these networks, researchers are connected if they have coauthored at least one paper together. The coauthored papers can then be interpreted as groups. The group structure induces a network: agents interact with other agents in their group, and the partial overlap of these groups gives rise to a network with a group structure. The agents are the vertices in the network, and the edges represent the relations between them, which are induced by the group structure. An example of this is shown in Figure 5.1, where the group structure is clearly visible.

The current chapter develops a random network model with a group structure. We refer to this model as the *random community model*. Each agent is assigned randomly to a subset of groups. A pair of agents is connected in the network if and only if they share at least one group. The probability with which an agent is assigned to groups depends on a random variable, the agent's *weight*. Agents with larger weights are more likely to belong to more groups and hence to acquire

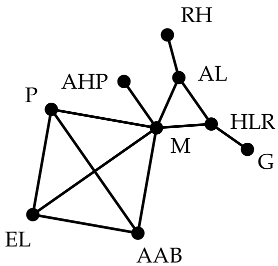


Figure 5.1. An example of an R&D network in the biotechnology sector. The figure displays all common R&D alliances of seven firms (Abbott Laboratories (AL), Astra AB (AAB), Eli Lilly & Co (EL), Hoffmann-La Roche Inc. (HLR), American Home Products Corp (AHP), Millennium (M), and Pfizer (P)) in the period 1994–2007. Other firms in the network are Genetics Institute Inc. (G), and Roche Holding AG (RH). Data are from the Thomson SDC Platinum database, and have been courteously provided by Bastian Westbrock.

a large number of contacts, i.e., to have a high degree. In the setting of social networks, the weight of an agent can be interpreted as a measure of the effort he invests in socializing. In the context of R&D networks, the weight of a firm could represent the amount it invests to become an attractive R&D partner, etcetera.

We characterize the degree distribution and the clustering of the random community model. The degree distribution and clustering are the stochastic analogues of the degree sequence and the clustering coefficient, which are important properties of networks.[1] Many social and economic networks have a power-law degree sequence, meaning that the fraction of agents with a given degree falls off roughly as a power law (see below for precise definitions). In that case, we say that the degree sequence is heavy-tailed. Moreover, many social and economic networks are characterized by a high clustering coefficient. Loosely speaking, the clustering coefficient of a network measures the number of closed triangles (sets of three agents each of which is connected to each of the others) in the network

[1] While the degree sequence of a network gives the fraction of vertices with a certain degree in a network, the degree distribution of a random network model specifies the probability that a vertex selected uniformly at random from a network has a certain degree. Similarly, the clustering coefficient of a network measures the number of closed triangles in a network relative to the number of connected triples, whereas the clustering of a random network model is the conditional probability that two vertices with a common neighbor are also direct neighbors.

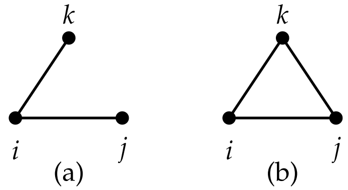


Figure 5.2. (a) A network with a single connected triple (with agent i as the central agent) and no closed triangles; (b) a network with one closed triangle, consisting of agents i, j, and k, and three connected triples (each with another agent as the central agent).

relative to the number of connected triples (a single agent with edges running to an unordered pair of others; see Figure 5.2).[2] In the context of social networks, if the clustering coefficient of a network is high, the friend of a friend is likely to be your friend. One reason why the clustering coefficient of social networks is typically high is indeed that people tend to interact in groups (Newman and Park, 2003).

The random community model we propose can account for these features of real networks. By choosing the parameters appropriately, we can control the tail behavior of the degree distribution in the model, and we can vary the clustering continuously. This allows us to obtain random network models with high clustering and a degree distribution with a power law tail. The clustering of the random community model follows from the group structure. Suppose all agents belong to a single group with probability one. In that case, the clustering would be equal to 1: all agents that have a common friend, are also directly connected. At the other extreme, suppose each agent belongs to a large number of groups. Then, two agents having a common neighbor, meaning that they share at least one group with this third agent, may well belong to different groups of this third agent, so that it is quite likely that they are not directly connected. In that case, the clustering of the random community model will be negligible. In the intermediate cases, in which there is a partial overlap of groups, as in Figure 5.1, the clustering will be between 0 and 1. The degree distribution of the random community model follows from the distribution of the weights of the agents. We show that if the distribution function of the weights is a power law with a given exponent, then the degree distribution has power law tails as well.

2 Also other definitions of the clustering coefficient exist in the literature, but the main idea behind these definitions is the same; see e.g. Newman (2003b).

The motivation for this model is twofold. Firstly, the current random network model offers a flexible framework to model players' beliefs over the group structure in network games with incomplete information on the network structure. As discussed in the previous two chapters, random network models provide a means to model players' beliefs over the network when there is incomplete information about the network structure. The current model allows one to choose a suitable degree distribution and to continuously vary the clustering of the network. This allows for a study of the effect of players' beliefs on the group structure on game-theoretic predictions. The previous two chapters have argued that it is not just the degree distribution of a random network model that is important for game-theoretic predictions, but that local dependencies between the degrees of agents also play a role. The clustering of a random network model is one aspect of these local dependencies. The second motivation for this model is that it is a very natural model for many social and economic settings, given the prevalence of groups and communities in such settings. The main parameter, the weight of an agent, has a clear and intuitive interpretation: it can be seen as the agent's investment in forming connections.

The random community model we propose is a generalization of the so-called random intersection graph model.[3] The random intersection graph model was introduced by Singer (1995) and Karoński et al. (1999), and has been further studied and generalized in Fill et al. (2000), Godehardt and Jaworski (2002), Stark (2004) and Jaworski et al. (2006). Newman (2003a) and Newman and Park (2003) discuss a model that is closely related to the model of Godehardt and Jaworski (2002) and Jaworski et al. (2006).[4] Unlike the model proposed in the current chapter, the original random intersection graph model cannot account for the heavy-tailed degree sequences observed in many real-world networks (Stark, 2004, Thm. 2). By contrast, the current framework allows us to obtain random network models in which the tail behavior of the degree distribution can be tuned.

The difference between the current model and other random intersection graph models that can account for power law degree distributions such as the model of Godehardt and Jaworski (2002), is that rather than positing a distribution of number of groups per agent, we derive this distribution from the distribution of agents' weights. The advantage of this approach is twofold. Firstly, it allows for a behavioral explanation of the distribution of the number of groups per agent,

3 The terms "graph" and "network" have the same meaning in the current context and can be used interchangeably.

4 There is also a related literature on random hypergraph models which focuses mostly on networks in which the groups have a fixed size, see Bollobás (2001) for an overview.

e.g. in terms of heterogeneity in terms of preferences, as in Cabrales et al. (2007). Secondly, the current model is particularly tractable, allowing us to give a complete characterization of its most important properties as the clustering and the degree distribution. The current model is inspired by a generalization of the Erdős-Rényi random network models, the generalized random graph model of Britton et al. (2006) (also see Bollobás et al., 2007). In that model, agents are also endowed with weights, but these weights determine agents' propensity to form bilateral connections, rather than their propensity to join groups, as in the current model.

Yao et al. (2005) discuss a model closely related to ours. The main difference between their work and the work presented here is that Yao et al. (2005) only consider power-law degree distributions, while we allow for a wide class of degree distributions, and that they do not characterize the clustering of their model. Furthermore, in their model, also the groups are endowed with weights, so that the probability that an agent joins a group depends on the weights of both the agent and the group. The weight of a group could for instance represent its intrinsic attractiveness. Our model can be adapted to allow for this, but for game-theoretic purposes, the current approach in which only agents are endowed with weights seems to be more natural.

This chapter is organized as follows. The random community model is introduced in Section 5.2. Section 5.3 presents the main results on the degree distribution and the clustering of the model. In Section 5.4, we analyze the clustering for the important example of a power law weight distribution. Section 5.5 concludes.

5.2 The random community model

In this section, we define the random community model. Let $n \in \mathbb{N}$, and let $\alpha, \beta > 0$. Define

$$m := \lfloor \beta n^{\alpha} \rfloor. \tag{5.1}$$

Let $V^{(n)} = \{1, \ldots, n\}$ be a set of *n agents* and let $C^{(m)}$ be a set of *m groups* or communities. Let F be a cumulative distribution function with support in $\mathbb{R}_+ := [0, \infty)$, and let μ_F be the Lebesgue-Stieltjes measure associated with F (Section 2.2.3). For normalization purposes, we assume that a random variable with cumulative distribution function F has mean 1 if its mean is finite. Let

$$\mathcal{V} := \bigcup_{n \in \mathbb{N}} V^{(n)} = \mathbb{N},$$

and let $W_1, W_2, \ldots$ be a random sequence in $\mathbb{R}_+$, with each W_ℓ, $\ell \in \mathcal{V}$, independently distributed with distribution function F. For $i \in \mathcal{V}$, we refer to the random variable W_i as the *weight* of agent i.

We now define the relevant probability space. For $n \in \mathbb{N}$, let $(U_{i,c}^{(n)})_{i \in V^{(n)}, c \in C^{(m)}}$ be a sequence of independent random variables uniformly distributed on $[0,1]$, and define

$$\mathcal{U} := \bigcup_{\substack{n \in \mathbb{N}, \\ i \in V^{(n)}, c \in C^{(m)}}} U_{ic}^{(n)}.$$

Let

$$\mathcal{Y} := \mathcal{U} \cup \{W_1, W_2, \ldots\}.$$

Let $(X_t)_{t \in T}$ be a countable collection of random variables such that for each $t \in T$, there is exactly one $Y \in \mathcal{Y}$ such that $X_t = Y$. For $t \in T$, let $(\Omega_t, \mathscr{F}_t)$ be the measurable space to which X_t maps. That is, if we let $\mathscr{B}_+$ and $\mathscr{B}_{[0,1]}$ denote the restriction of the Borel σ-algebra to $\mathbb{R}_+$ and $[0,1]$, respectively, $(\Omega_t, \mathscr{F}_t)$ is equal to $(\mathbb{R}_+, \mathscr{B}_+)$ if $X_t = W_i$ for some $i \in \mathcal{V}$, and to $([0,1], \mathscr{B}_{[0,1]})$ otherwise. For each $t \in T$, let $\mathbb{P}_t$ be the distribution of X_t.

Let $\Omega := \times_{t \in T} \Omega_t$. We define the infinite product σ-algebra $\mathscr{F}$ on Ω in the standard way (e.g. Aliprantis and Border, 1999, Sec. 14.6). For each finite subset S of T, define

$$\mathscr{C}_S := \{\times_{t \in S} E_t \mid \forall t \in S, E_t \in \mathscr{F}_t\},$$

and let $\mathscr{F}_S := \sigma(\mathscr{C}_S)$ be the σ-algebra generated by $\mathscr{C}_S$. Then, for any $E = \times_{t \in S} E_t \in \mathscr{C}_S$, define

$$\mathbb{P}_S(E) := \prod_{t \in S} \mathbb{P}_t(E_t),$$

and extend $\mathbb{P}_S$ to $\mathscr{F}_S$ in the usual way (e.g. Billingsley, 1995).

Then, by Kolmogorov's extension theorem (e.g. Aliprantis and Border, 1999, Sec. 14.6), there is a unique probability measure $\mathbb{P}$ on the σ-algebra $\mathscr{F}$ in Ω such that for each finite subset S of T,

$$\mathbb{P} \circ \pi_{ST}^{-1} = \mathbb{P}_S,$$

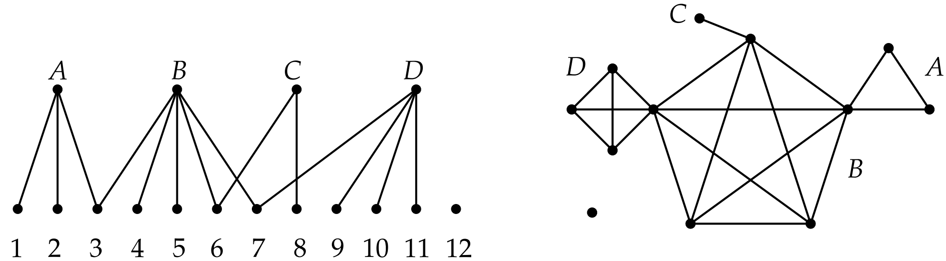


Figure 5.3. We construct the network in the right panel from the bipartite network with agent set $\{1, 2, \ldots, 12\}$ and group set $\{A, B, C, D\}$ in the left panel by drawing an edge between two distinct agents if and only if they share at least one group in the bipartite network. For clarity, only the group labels in the right panel are indicated.

where π_{ST} is the natural projection of Ω on $\times_{t \in S} \Omega_t$. In particular, for any $i \in \mathcal{V}$, for any $B \in \mathscr{B}_+$,

$$\mathbb{P}\Big(\big(W_i\big)^{-1}(B)\Big) = \mu_F(B).$$

Similarly, for any $n \in \mathbb{N}, i \in V^{(n)}, c \in C^{(m)}$, for any $B \in \mathscr{B}_{[0,1]}$,

$$\mathbb{P}\Big(\big(U_{i,c}^{(n)}\big)^{-1}(B)\Big) = \mu_u(B),$$

where μ_u is the Lebesgue-Stieltjes measure associated with the uniform distribution on $[0, 1]$.

We can now define the random community model. Let $\gamma > 0$, and let $n \in \mathbb{N}$. For $i \in V^{(n)}$, define

$$P_i^{(n)} := \min\Big\{\gamma W_i \, n^{-(1+\alpha)/2}, 1\Big\}. \tag{5.2}$$

Notice that we can ignore the min-operator in Equation (5.2) for a given value of an agent's weight when n is sufficiently large. Let $\mathcal{G}_B^{(n,m)}$ be the set of bipartite networks with independent sets $V^{(n)}$ and $C^{(m)}$, and let $\mathscr{F}_B^{(n,m)}$ be the set of subsets of $\mathcal{G}_B^{(n,m)}$. Define the $\mathscr{F}/\mathscr{F}_B^{(n,m)}$-measurable function $B_F^{(n,m)}$ as follows. For each $\omega \in \Omega$, $B_F^{(n,m)}(\omega)$ is the bipartite network that is obtained by adding an edge between agent $i \in V^{(n)}$ and group $c \in C^{(m)}$ if and only if

$$P_i^{(n)}(\omega) \geq U_{i,c}^{(n)}(\omega).$$

This means that there is an edge between i and c with probability $P_i^{(n)}(\omega)$, independent of other edges given W_i. Denote the law of $B_F^{(n,m)}$ by $\mathbb{P}_{F,B}^{(n,m)}$. We refer to $B_F^{(n,m)}$ as the *random bipartite network*, and to $(\mathcal{G}_B^{(n,m)}, \mathscr{F}_B^{(n,m)}, \mathbb{P}_{F,B}^{(n,m)})$ as the *random bipartite network model* with agent set $V^{(n)}$ and group set $C^{(m)}$. The following definition will be useful:

$$\forall E \in \mathscr{F}_B^{(n,m)}: \quad \hat{\mathbb{P}}_{F,B}^{(n,m)}(E) := \mathbb{P}\left(\left\{\omega \in \Omega \mid B_F^{(n,m)}(\omega) \in E\right\} \,\Big|\, \{W_i\}_{i \in V^{(n)}}\right).$$

That is, $\hat{\mathbb{P}}_{F,B}^{(n,m)}$ is the conditional law of $B_F^{(n,m)}$. Notice that $\hat{\mathbb{P}}_{F,B}^{(n,m)}(E)$ is a random variable for all $E \in \mathscr{F}_B^{(n,m)}$ (Section 2.2.4).

From any bipartite network $g_B \in \mathcal{G}_B^{(n,m)}$ with independent sets $V^{(n)}$ and $C^{(m)}$, we can obtain a network with vertex set $V^{(n)}$ by drawing an edge between two agents $i, j \in V^{(n)}, i \neq j$, if and only if they have a common adjacent vertex $c \in C^{(m)}$ in g_B, as illustrated in Figure 5.3. Let $\mathcal{G}_G^{(n,m)}$ be the set of all networks with n vertices that can be obtained in this way, and let $\mathscr{F}_G^{(n,m)}$ be the set of all subsets of $\mathcal{G}_G^{(n,m)}$. This defines the $\mathscr{F}/\mathscr{F}_G^{(n,m)}$-measurable function $G_F^{(n,m)}$, referred to as the *random community network*. Denote the law of $G_F^{(n,m)}$ by $\mathbb{P}_{F,G}^{(n,m)}$. Then, $(\mathcal{G}_G^{(n,m)}, \mathscr{F}_G^{(n,m)}, \mathbb{P}_{F,G}^{(n,m)})$ is the *random community model*. As before, define

$$\forall E \in \mathscr{F}_G^{(n,m)}: \quad \hat{\mathbb{P}}_{F,G}^{(n,m)}(E) := \mathbb{P}\left(\left\{\omega \in \Omega \mid G_F^{(n,m)}(\omega) \in E\right\} \,\Big|\, \{W_i\}_{i \in V^{(n)}}\right),$$

i.e., $\hat{\mathbb{P}}_{F,G}^{(n,m)}$ is the conditional law of $G_F^{(n,m)}$. Again, $\hat{\mathbb{P}}_{F,G}^{(n,m)}(E)$ is a random variable for all $E \in \mathscr{F}_G^{(n,m)}$.[5]

Hence, in a realization of the random community network, vertices represent the agents, and the edges represent the relations among agents. The groups determine the network structure. Notice that there is "double randomness" in the model: first we draw the weights of agents according to some distribution function, and then agents are randomly assigned to groups, with the probability with which agents are assigned to groups being determined by their weights.

Remark 5.2.1. In the definition of the random community model, we use the terms "agents" and "groups", with the understanding that the model is of course much more general; see Section 5.1 for some examples in economics, and see Palla et al. (2005) for examples in other fields. ◃

5 The random bipartite network, the random community network and the associated random network models are defined for given α, β and γ; to simplify notation, indices α, β and γ are suppressed.

Remark 5.2.2. In the current model, weights are assigned at random according to some distribution function F (which could be degenerate, i.e., assign weight $w \in \mathbb{R}_+$ with probability 1). This distribution could follow from the strategic choices of agents or players. For instance, in Cabrales et al. (2007), who apply a random network model that is similar to the current model, each player has to choose a networking investment. The networking investment of a player determines the probability with which he forms bilateral links with other players, just like the weight determines the probability with which an agent joins groups in our model. A player's networking investment is a strategic choice in the model of Cabrales et al. (2007): it depends on a player's preferences and the choices of other players. Heterogeneity in preferences leads to a non-degenerate distribution function of networking investments and thus to heterogeneous networks. Also see the discussion in Section 5.5. ◃

Remark 5.2.3. The particular functional form (5.1) for the number of groups is chosen in order to obtain an interesting class of random network models; see Karoński et al. (1999) for a discussion. Intuitively, the number of groups has to grow with the number of agents. If it were to remain constant as the number of agents grows, then, asymptotically, given the functional form of $P_i^{(n)}$, $i \in V^{(n)}$, the expected number of groups per agent would be zero, so that the network would be empty. At the other extreme, suppose that the number of groups would grow much faster than the number of agents. In that case, agents would belong to a large number of groups, but often, these groups contain at most one other agent, so that there is no nontrivial group structure. These intuitions are reflected in our main results on the degree distribution (Theorem 5.3.2) and clustering (Theorem 5.3.4) of the random community network model, see below for a discussion. ◃

We are interested in the properties of the model when $n \to \infty$, in particular the degree distribution and the clustering, which we define now. Recalling that there is a fixed relationship (5.1) between n and m for given α and β, define

$$\mathcal{G}_B := \bigcup_{n \in \mathbb{N}} \mathcal{G}_B^{(n,m)}.$$

Let $\mathscr{F}_B$ be the σ-algebra generated by the set of singletons of $\mathcal{G}_B$. Define $\mathcal{G}_G$ and $\mathscr{F}_G$ analogously to $\mathcal{G}_B$ and $\mathscr{F}_B$, respectively. Let $\mathcal{Q}$ be the set of all finite subsets of $\mathcal{V}$, and let $V : \mathcal{G}_G \to \mathcal{Q}$ and $V_B : \mathcal{G}_B \to \mathcal{Q}$ be the functions that assign to each $g \in \mathcal{G}_G$, respectively each $g_B \in \mathcal{G}_B$, its agent set. That is, if $g \in \mathcal{G}_G^{(n,m)}$ for some $n \in \mathbb{N}$, then $V(g) = V^{(n)}$, and if $g_B \in \mathcal{G}_B^{(n,m)}$, then $V_B(g_B) = V^{(n)}$. By definition, for each $g \in \mathcal{G}_G$ there is at least one corresponding bipartite network $g_B \in \mathcal{G}_B$, but there may be more than one. For each $g \in \mathcal{G}_G$, denote by $B(g) \subseteq \mathcal{G}_B$ the set of bipartite networks that correspond to it.

For $i \in \mathcal{V}$, let D_i be the function on $\mathcal{G}_G$ that assigns to each $g \in \mathcal{G}_G$ the number of neighbors of i in g if $i \in V(g)$ and zero otherwise. For each $n \in \mathbb{N}$, define $D_i(G_F^{(n,m)})$ by

$$\forall \omega \in \Omega : \quad D_i\Big(G_F^{(n,m)}\Big)(\omega) := D_i\left(G_F^{(n,m)}(\omega)\right).$$

It can be easily checked that $D_i(G_F^{(n,m)})$ is a random variable. We refer to $D_i(G_F^{(n,m)})$ as the *degree of i in $G_F^{(n,m)}$*. We now derive a convenient expression for the degree of an agent in a random community network. For $i, j \in \mathcal{V}, i \neq j$, let Z_{ij} be the function on $\mathcal{G}_B$ that assigns to each $g_B \in \mathcal{G}_B$ the number of groups that agent i and agent j have in common in g_B if $i, j \in V_B(g_B)$, and zero otherwise. As before, for $n \in \mathbb{N}$, define $Z_{ij}(B_F^{(n,m)})$ by

$$\forall \omega \in \Omega : \quad Z_{ij}\Big(B_F^{(n,m)}\Big)(\omega) := Z_{ij}\left(B_F^{(n,m)}(\omega)\right).$$

Again, it is straightforward to verify that $Z_{ij}(B_F^{(n,m)})$ is a random variable. Since two agents are connected in a network if and only if they share at least one group in the corresponding bipartite network, it holds that

$$\forall g \in \mathcal{G}_G, \forall g_B, g_B' \in B(g), \forall i, j \in \mathcal{V}, i \neq j : \quad Z_{ij}(g_B) \geq 1 \iff Z_{ij}(g_B') \geq 1,$$

and we can write

$$\forall \omega \in \Omega : \quad D_i\Big(G_F^{(n,m)}\Big)(\omega) = \sum_{j \in \mathcal{V} \setminus \{i\}} \mathbf{1}_{[Z_{ij}(B_F^{(n,m)}) \geq 1]}(\omega), \tag{5.3}$$

where we recall that $\mathbf{1}_E$ is the indicator function of the event E (see Section 2.2.2). Finally, let N_c be the function that assigns to each $g_B \in \mathcal{G}_B$ the number of groups agent 1 belongs to. For each $n \in \mathbb{N}$, define $N_c(B_F^{(n,m)})$ by:

$$\forall \omega \in \Omega : \quad N_c\Big(B_F^{(n,m)}\Big)(\omega) := N_c\left(B_F^{(n,m)}(\omega)\right).$$

Then, $N_c(B_F^{(n,m)})$ is a random variable defined on Ω.

The *degree distribution* of the random community model $(\mathcal{G}_G^{(n,m)}, \mathscr{F}_G^{(n,m)}, \mathbb{P}_{F,G}^{(n,m)})$, or equivalently, of the random community network $G_F^{(n,m)}$, gives for each $t \in \mathbb{N}_0$ the probability that an agent selected uniformly at random from the network has degree t (also see Section 2.3.1). A degree distribution with associated cumulative distribution function H is *heavy-tailed* or has *heavy tails* if for all $a > 0$

$$\lim_{t \to \infty} e^{at}(1 - H(t)) \to \infty.$$

A degree distribution is a *power law* if the associated cumulative distribution function H satisfies $(1 - H(t)) \sim t^{-\alpha}$, where $\alpha > 0$ (recall that $\sim$ denotes that two functions are asymptotically equal, see page xv for a precise definition). A power law degree distribution is a primary example of a heavy-tailed degree distribution.

We also study the clustering of random network models. While the degree distribution is a purely local property of a random network model, referring solely to the connections of a single agent, the clustering is a property that refers to the connections of an agent and his neighbors. Informally, the clustering of a random network model is the probability that there is an edge between two vertices given that they have a common neighbor, as illustrated in Figure 5.4. More formally, let $n \in \mathbb{N}$, and for $i, j \in V^{(n)}, i \neq j$, denote the event (in $\mathscr{F}_B^{(n,m)}$) that agents i and j have a common group in the random bipartite network by $E_{ij}^{(n)}$ (note that this is equivalent to the event (in $\mathscr{F}_G^{(n,m)}$) that there is an edge between i and j in the random community network). Let $i, j, k \in V^{(n)}$ be three distinct agents. Then, the clustering in the random community model with n agents is defined as

$$c_F^{(n,m)} := \mathbb{E}\left[\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ij}^{(n)} \mid E_{ik}^{(n)}, E_{jk}^{(n)}\right)\right], \tag{5.4}$$

where the expectation is taken over the weights. Clearly, the agents in $V^{(n)}$ are indistinguishable, so that $c_F^{(n,m)}$ does not depend on the particular choice of i, j and k. Furthermore, define

$$c_F := \lim_{n\to\infty} c_F^{(n,m)}$$

to be the *(asymptotic) clustering* of the random community model.

Remark 5.2.4. Other definitions for the clustering in random network models exist. For instance, one could look at the expected ratio of closed triangles to the number of connected triples. Alternatively, one could consider the expectation of the ratio of the number of triangles (sets of three agents each of which is connected to each of the others) connected to a vertex relative to the number of connected triples (a single agent with edges running to an unordered pair of others) centered on that vertex, averaged over all vertices (e.g. Jackson, 2008). Our definition should give similar results as the first definition; further research is needed to determine the exact relation between the different clustering definitions in random network models. ◃

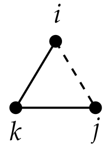


Figure 5.4. The clustering of a random network model is the conditional probability that two arbitrary vertices i, j are neighbors given that they have a common neighbor k.

5.3 Analysis

In this section, we characterize the (asymptotic) degree distribution and the (asymptotic) clustering in the random community model as a function of the parameters α, β and γ and the distribution function F for the weights. All unspecified limits are taken as $n \to \infty$. We first characterize the degree distribution and then proceed to the clustering.

5.3.1 Degree distribution

Let $n \in \mathbb{N}$, and let $i, j \in V^{(n)}, i \neq j$. Conditional on the weights W_i and W_j, the probability that there is an edge between i and j in the random community network is

$$1 - \left(1 - P_i^{(n)} P_j^{(n)}\right)^m = \beta\gamma^2 W_i W_j n^{-1} + O\left(W_i^2 W_j^2\, n^{-2}\right).$$

By summing the expectations of the edge indicators over j, it is easy to see that, at least when the weights have finite second moment, the expected degree of agent i given his weight W_i is asymptotically $\beta\gamma^2 W_i$ (recall that if W_i has finite mean, we assume that $\mathbb{E}[W_i] = 1$). Theorem 5.3.2 below, which is a generalization of Theorem 2 of Stark (2004), gives a full characterization of the degree distribution for different values of the parameters. Theorem 5.3.2 uses Lemma 5.3.1, the proof of which can be found in Appendix 5.B.

Lemma 5.3.1. *Fix $\eta > 0$. If W_1 has finite first moment, then, conditional on the weight W_1 of agent 1, for any $\delta > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\left(N_c(B_F^{(n,m)})\right)^2 \sum_{j\in V^{(n)}} \left(P_j^{(n)}\right)^2 > \eta\right) \leq \delta W_1,$$

which implies

$$\left(N_c(B_F^{(n,m)})\right)^2 \sum_{j\in V^{(n)}} \left(P_j^{(n)}\right)^2 \xrightarrow{\mathbb{P}} 0.$$

Theorem 5.3.2. *Let $n \in \mathbb{N}$. Consider the random community network $G_F^{(n,m)}$ with $m = \lfloor \beta n^\alpha \rfloor$ and $P_i^{(n)}$, $i \in V^{(n)}$, as in* (5.2).

(a) If $\alpha < 1$ and if W_1 has finite first moment, then the probability mass function of the degree $D_i(G_F^{(n,m)})$ of agent $i \in V^{(n)}$ in the random community network $G_F^{(n,m)}$ converges to a point mass (of weight 1) at 0 as $n \to \infty$.

(b) If $\alpha = 1$ and if W_1 has finite first moment, then the degree $D_i(G_F^{(n,m)})$ of an agent i in the random community network $G_F^{(n,m)}$ with weight W_i converges in distribution to a sum of a Poisson($\beta\gamma W_i$) distributed number of Poisson(γ) variables as $n \to \infty$, where all variables are independent. That is, $D_i(G_F^{(n,m)})$ converges in distribution to a random variable with a compound Poisson distribution.

(c) If $\alpha > 1$ and if W_1 has finite first moment, then the degree $D_i(G_F^{(n,m)})$ of an agent i in the random community network with weight W_i is asymptotically Poisson($\beta\gamma^2 W_i$) distributed.

Proof. We prove the theorem for agent $i = 1$. As agents are indistinguishable, this proves the theorem for all agents.

(a): If there is no group to which agent 1 belongs, then clearly his degree is 0. Hence, it is sufficient to show that when $\alpha < 1$,

$$\mathbb{P}\left(N_c(B_F^{(n,m)}) = 0\right) \to 1$$

as $n \to \infty$. Let $n \in \mathbb{N}$. Conditional on W_1, the random variable $N_c(B_F^{(n,m)})$ is binomially distributed with parameters m and $P_1^{(n)}$. Consequently,

$$\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(\left\{g_B \in \mathcal{G}_B^{(n,m)} \;\middle|\; N_c(g_B) = 0\right\}\right) \geq \left(1 - P_1^{(n)}\right)^m = 1 - O\left(m\, P_1^{(n)}\right).$$

By the definitions of m and $P_1^{(n)}$, we have that $m\, P_1^{(n)} \leq \beta\gamma W_1 n^{(\alpha-1)/2}$. By Markov's inequality (Lemma 2.2.34),

$$\forall \eta > 0 : \quad \mathbb{P}\left(W_1 n^{(\alpha-1)/2} > \eta\right) \leq \frac{1}{\eta\, n^{(1-\alpha)/2}}\, \mathbb{E}[W_1].$$

If $\alpha < 1$ and $\mathbb{E}[W_1] < \infty$, then the right-hand side of the expression above converges to 0. It follows that

$$\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(\left\{g_B \in \mathcal{G}_B^{(n,m)} \;\middle|\; N_c(g_B) = 0\right\}\right) \xrightarrow{\mathbb{P}} 1$$

By bounded convergence (Lemma 2.2.36), we obtain that

$$\mathbb{P}\left(N_c(B_F^{(n,m)}) = 0\right) = \mathbb{E}\left[\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(\left\{g_B \in \mathcal{G}_B^{(n,m)} \;\middle|\; N_c(g_B) = 0\right\}\right)\right] \to 1$$

as $n \to \infty$

(b, c): To prove (b) and (c), we show that the generating function of $D_1(G_F^{(n,m)})$ converges to the generating function of a random variable with distribution function as specified in (b) and (c), respectively (recall Theorem 2.2.31).

For ease of notation, write $Z_j := Z_{1j}(B_F^{(n,m)})$ and $N_n := N_c(B_F^{(n,m)})$. Conditional on N_n and $\{W_j\}_{j\in V^{(n)}\setminus\{1\}}$, the random variables $Z_2, \ldots, Z_n$ are independent and for each $j = 2, \ldots, n$, Z_j is binomially distributed with parameters N_n and $P_j^{(n)}$. Hence, using (5.3), we can write the probability generating function of $D_1(G_F^{(n,m)})$ (conditional on W_1) as

$$\begin{aligned}
&\mathbb{E}\left[t^{D_1(G_F^{(n,m)})} \;\middle|\; W_1\right] \\
&\quad = \mathbb{E}\left[\prod_{j\in V^{(n)}\setminus\{1\}} \mathbb{E}\left[t^{\mathbf{1}[Z_j\geq 1]} \;\middle|\; \{W_j\}_{j\in V^{(n)}}, N_n\right] \;\middle|\; W_1\right] \\
&\quad = \mathbb{E}\left[\prod_{j\in V^{(n)}\setminus\{1\}} \left(1 + (t-1)\mathbb{P}_{F,B}^{(n,m)}\left(Z_j \geq 1 \;\middle|\; \{W_j\}_{j\in V^{(n)}\setminus\{1\}}, N_n\right)\right) \;\middle|\; W_1\right],
\end{aligned}$$

where $t \in [0,1]$. Using the Taylor expansion $\log(1+x) = x + O(x^2)$ (for $x \downarrow 0$) and

$$\mathbb{P}_{F,B}^{(n,m)}\left(Z_j \geq 1 \;\middle|\; \{W_j\}_{j\in V^{(n)}\setminus\{1\}}, N_n\right) = 1 - \left(1 - P_j^{(n)}\right)^{N_n} = N_n\, P_j^{(n)} + O\left(N_n^2 \left(P_j^{(n)}\right)^2\right),$$

we obtain

$$\begin{aligned}
&\prod_{j\in V^{(n)}\setminus\{1\}} \left(1 + (t-1)\mathbb{P}_{F,B}^{(n,m)}\left(Z_j \geq 1 \;\middle|\; \{W_j\}_{j\in V^{(n)}\setminus\{1\}}, N_n\right)\right) \\
&\quad = \exp\left((t-1)\, N_n \sum_{j\in V^{(n)}\setminus\{1\}} P_j^{(n)} + O\left(N_n^2 \sum_{j\in V^{(n)}\setminus\{1\}} \left(P_j^{(n)}\right)^2\right)\right) \\
&\quad = e^{(t-1)\, N_n \sum_{j\in V^{(n)}\setminus\{1\}} P_j^{(n)}} + R_n,
\end{aligned} \tag{5.5}$$

where in the last line we have defined

$$R_n := \exp\left((t-1)\, N_n \sum_{j\in V^{(n)}\setminus\{1\}} P_j^{(n)}\right)\left(\exp\left(O\left(N_n^2 \sum_{j\in V^{(n)}\setminus\{1\}} \left(P_j^{(n)}\right)^2\right)\right) - 1\right).$$

For the remainder over the proof, all unspecified summations over j run over $V^{(n)} \setminus \{1\}$. Since the product in (5.5) is the conditional expectation of $t^{D_1(G_F^{(n,m)})}$ with $t \in [0,1]$, it takes values between 0 and 1, and, since $\exp((t-1)\, N_n \sum_j P_j^{(n)}) \in (0,1]$, it follows that $R_n \in [-1,1]$. We show that

$$(i) \qquad \mathbb{E}\left[e^{(t-1)N_n \sum_j P_j^{(n)}} \mid W_1\right] \overset{\text{a.s.}}{\to} \exp\left(\beta\gamma W_1(e^{\gamma(t-1)} - 1)\right), \qquad \text{if } \alpha = 1;$$

$$(ii) \qquad \mathbb{E}\left[e^{(t-1)N_n \sum_j P_j^{(n)}} \mid W_1\right] \overset{\text{a.s.}}{\to} \exp\left(\beta\gamma^2 W_1(t-1)\right), \qquad \text{if } \alpha > 1;$$

$$(iii) \qquad \mathbb{E}[R_n \mid W_1] \overset{\text{a.s.}}{\to} 0.$$

The limits in (i) and (ii) are the generating functions for random variables with the compound Poisson and Poisson distributions specified in part (b) and (c) of the theorem, respectively. Hence (recall Theorem 2.2.31), establishing (i)-(iii) proves part (b) and (c) of the theorem.

We start with (i). First notice that the expectation with respect to N_n of $\exp((t-1)\, N_n \sum_j P_j^{(n)})$ is given by the generating function for N_n evaluated at the point $\exp((t-1)\sum_j P_j^{(n)})$. Since N_n is binomially distributed with parameters m and $P_1^{(n)}$, we have that

$$\mathbb{E}\left[e^{(t-1)N_n \sum_j P_j^{(n)}} \,\Big|\, W_1\right] = \mathbb{E}\left[\left(1 + P_1^{(n)}\left(e^{(t-1)\sum_j P_j^{(n)}} - 1\right)\right)^m \,\Big|\, W_1\right]. \tag{5.6}$$

For $\alpha = 1$, we have $m = \lfloor \beta n \rfloor$ and $P_j^{(n)} = \min\{\gamma W_j n^{-1}, 1\}$ for each $j \in V^{(n)}$. Recall that $\mathbb{E}[W_j] = 1$ for all $j \in V^{(n)}$ if W_j has a finite mean. Hence, by the strong law of large numbers (Theorem 2.2.33),

$$\sum_{j \in V^{(n)} \setminus \{1\}} P_j^{(n)} \overset{\text{a.s.}}{\to} \gamma.$$

Since we condition on W_1,

$$\left(1 + P_1^{(n)}\left(e^{(t-1)\sum_j P_j^{(n)}} - 1\right)\right)^{\lfloor \beta n \rfloor} \overset{\text{a.s.}}{\to} \exp\left(\beta\gamma W_1(e^{\gamma(t-1)} - 1)\right)$$

as $n \to \infty$. Noting that $t \in [0,1]$, it then follows from the theorem of bounded convergence for conditional expectations (Billingsley, 1995, Sec. 34) that

$$\mathbb{E}\left[\left(1 + P_1^{(n)}\left(e^{(t-1)\sum_j P_j^{(n)}} - 1\right)\right)^{\lfloor \beta n \rfloor}\right] \overset{\text{a.s.}}{\to} \exp\left(\beta\gamma W_1(e^{\gamma(t-1)} - 1)\right) \quad \text{as } n \to \infty,$$

proving (i).

To show (*ii*), let $j \in V^{(n)}$, and for $\alpha > 1$, define $q_j := n^{(\alpha-1)/2} P_j^{(n)}$. Recall that $m = \lfloor \beta n^\alpha \rfloor$ and that $P_j^{(n)} = \min\{\gamma W_j n^{-(1+\alpha)/2}, 1\}$ for each $j \in V^{(n)}$. We assume that n is sufficiently large so that $\gamma W_1 n^{-(1+\alpha)/2} \le 1$. After some rewriting, we obtain

$$\left(1 + P_1^{(n)} \left(e^{(t-1)\sum_j P_j^{(n)}} - 1\right)\right)^m = \left(1 + \left(\frac{\gamma W_1 (t-1) \sum_j q_j}{n^\alpha}\right) \cdot \left(\frac{e^{(t-1)n^{(1-\alpha)/2} \sum_j q_j} - 1}{(t-1) n^{(1-\alpha)/2} \sum_j q_j}\right)\right)^{\lfloor \beta n^\alpha \rfloor}. \quad (5.7)$$

By the strong law of large numbers, $\sum_j q_j \xrightarrow{\text{a.s.}} \gamma$. Since $(e^{ax} - 1)/x \to a$ for $a \in \mathbb{R}_+$ as $x \to 0$, it follows that the right-hand side of (5.7) converges almost surely to $\exp(\beta\gamma^2 W_1 (t-1))$ as $n \to \infty$. By (5.6) and bounded convergence, this proves (*ii*).

It remains to prove (*iii*). By Lemma 5.3.1,

$$N_n^2 \sum_{j \in V^{(n)} \setminus \{1\}} \left(P_j^{(n)}\right)^2 \xrightarrow{\text{p}} 0.$$

Since $t \in [0, 1]$, this implies that R_n converges in probability to 0, so that (*iii*) then follows from bounded convergence. □

Theorem 5.3.2 states that, depending on the value of α, the degree distribution converges to a point mass at 0 (for $\alpha < 1$), a compound Poisson distribution (for $\alpha = 1$) or a Poisson distribution (for $\alpha > 1$). Importantly, this result means that for $\alpha = 1$, we can tune the tail behavior of the degree distribution. In particular, as demonstrated in Appendix 5.A, we can obtain a degree distribution with power law tails by choosing the weight distribution to be a power law. Intuitively, the distribution of the summands in part (b) of the theorem has so-called thin tails, while the distribution of the number of summands will inherit the heavy tails of the weight distribution. The heavy tails of the latter distribution will then dominate the tail behavior of the degree distribution.

To get some intuition for Theorem 5.3.2, note that the expected number of groups that agent $i \in V^{(n)}$ belongs to is roughly $\beta\gamma W_i n^{(\alpha-1)/2}$. If $\alpha < 1$ and if W_i has finite mean, this number converges in probability to 0, and hence, the degree of an agent in a random community network converges in probability to 0 in that case, as stated in part (a) of the theorem. For $\alpha = 1$, the number of groups that agent i is a member of is Poisson($\beta\gamma W_i$) distributed as $n \to \infty$, and the number of other agents in each of these groups is approximately Poisson(γ) distributed, which explains part (b) of the theorem. Finally, for $\alpha > 1$, an agent belongs to infinitely many groups as $n \to \infty$. This means that the edge indicators will be asymptotically independent, giving rise to the Poisson distribution as specified in part (c) of the theorem.

5.3.2 Clustering

To characterize the asymptotic clustering of the random community model, we need some more notation. Let $n \in \mathbb{N}$. For three distinct agents $i, j, k \in V^{(n)}$, denote by $E^{(n)}_{ijk}$ the event (in $\mathscr{F}^{(n,m)}_B$) that there is at least one group in the random bipartite network to which all three agents i, j and k belong, and write $E^{(n)}_{ij,ik,jk}$ for the event that there are at least three *distinct* groups to which i and j, i and k, and j and k respectively belong. Similarly, the event that there are two distinct groups to which agents i and k, and j and k respectively belong is denoted by $E^{(n)}_{ik,jk}$. Notice that both events $E^{(n)}_{ijk}$ and $E^{(n)}_{ij,ik,jk}$ are equivalent to the event in $\mathscr{F}^{(n,m)}_G$ that there is an edge between i and j, between j and k and between i and j in the random community network.

We use the following lemma:

Lemma 5.3.3. *Let $n \in \mathbb{N}$. Consider the random community network $G^{(n,m)}_F$ with $m = \lfloor \beta n^\alpha \rfloor$ and $P^{(n)}_i$, $i \in V^{(n)}$, as in* (5.2). *For any three distinct agents $i, j, k \in V^{(n)}$,*

(a) $\hat{\mathbb{P}}^{(n,m)}_{F,B}\left(E^{(n)}_{ijk}\right) = \frac{\beta\gamma^3 W_i W_j W_k}{n^{(3+\alpha)/2}} + O\left(\frac{W_i^2 W_j^2 W_k^2}{n^{3+\alpha}}\right)$;

(b) $\hat{\mathbb{P}}^{(n,m)}_{F,B}\left(E^{(n)}_{ij,ik,jk}\right) = \frac{\beta^3\gamma^6 W_i^2 W_j^2 W_k^2}{n^3} + O\left(\frac{W_i^3 W_j^3 W_k^3}{n^4}\right)$;

(c) $\hat{\mathbb{P}}^{(n,m)}_{F,B}\left(E^{(n)}_{ik,jk}\right) = \frac{\beta^2\gamma^4 W_i W_j W_k^2}{n^2} + O\left(\frac{W_i^2 W_j^2 W_k^3}{n^3}\right)$;

(d) $\hat{\mathbb{P}}^{(n,m)}_{F,B}\left(E^{(n)}_{ijk} E^{(n)}_{ik,jk}\right) = O\left(\frac{W_i^2 W_j^2 W_k^2}{n^{(5+\alpha)/2}}\right)$.

The proof can be found in Appendix 5.B.

We are now ready to state and prove Theorem 5.3.4.

Theorem 5.3.4. *Let $n \in \mathbb{N}$. Consider the random community network $G^{(n,m)}_F$ with $m = \lfloor \beta n^\alpha \rfloor$ and $P^{(n)}_i$, $i \in V^{(n)}$, as in* (5.2). *If W_1 has finite mean,*

(a) $c_F = 1$ *for* $\alpha < 1$;

(b) $c_F = \mathbb{E}\left[(1 + \beta\gamma W_k)^{-1}\right]$ *for* $\alpha = 1$;

(c) $c_F = 0$ *for* $\alpha > 1$.

Proof. Note that for three distinct agents $i, j, k \in V^{(n)}$,

$$\hat{\mathbb{P}}^{(n,m)}_{F,B}\left(E^{(n)}_{ij} \,\middle|\, E^{(n)}_{ik} \cap E^{(n)}_{jk}\right) = \frac{\hat{\mathbb{P}}^{(n,m)}_{F,B}\left(E^{(n)}_{ijk} \cup E^{(n)}_{ij,ik,jk}\right)}{\hat{\mathbb{P}}^{(n,m)}_{F,B}\left(E^{(n)}_{ijk} \cup E^{(n)}_{ik,jk}\right)}.$$

(a): Applying Lemma 5.3.3 and combining the error terms yields

$$\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ij}^{(n)} \,\middle|\, E_{ik}^{(n)} \cap E_{jk}^{(n)}\right) \geq \frac{\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ijk}^{(n)}\right)}{\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ijk}^{(n)}\right) + \hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ik,jk}^{(n)}\right)}$$

$$= \frac{1 + O\left(W_i W_j W_k \, n^{-(3+\alpha)/2}\right)}{1 + W_k\left[\beta\gamma n^{(\alpha-1)/2} + O\left(W_i W_j W_k \, n^{-(3-\alpha)/2}\right)\right]}. \tag{5.8}$$

By Markov's inequality, when $\alpha < 1$,

$$\mathbb{P}\left(W_i W_j W_k \, n^{-(3-\alpha)/2} > \eta\right) \leq \frac{1}{\eta n}\,\mathbb{E}[W_i W_j W_k] \quad \text{for any } \eta > 0.$$

Hence, since W_i, W_j and W_k are independent and have finite mean,

$$\mathbb{P}\left(W_i W_j W_k \, n^{-(3-\alpha)/2} > \eta\right) \to 0.$$

Consequently, $W_i W_j W_k \, n^{-(3-\alpha)/2}$ converges in probability to 0 when $n \to \infty$. Similarly,

$$W_i W_j W_k \, n^{-(3+\alpha)/2} \xrightarrow{\mathbb{P}} 0.$$

Furthermore, it is easy to see that for $\alpha < 1$,

$$n^{(\alpha-1)/2} \to 0$$

as $n \to \infty$. Hence, the fraction in (5.8) converges in probability to 1 for $\alpha < 1$. Hence, by bounded convergence $c_F^{(n,m)} \to 1$ for $\alpha < 1$.

(b): First note that for $\alpha = 1$, it follows from (5.8) and the above reasoning that

$$\liminf c_F^{(n,m)} \geq \mathbb{E}\left[(1 + \beta\gamma W_k)^{-1}\right].$$

Hence, it remains to show that

$$\limsup c_F^{(n,m)} \leq \mathbb{E}\left[(1 + \beta\gamma W_k)^{-1}\right].$$

Applying Lemma 5.3.3 with $\alpha = 1$ and simplifying yields

$$\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ij}^{(n)} \,\middle|\, E_{ik}^{(n)} \cap E_{jk}^{(n)}\right) \leq \frac{\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ijk}^{(n)}\right) + \hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ij,ik,jk}^{(n)}\right)}{\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ijk}^{(n)}\right) + \hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ik,jk}^{(n)}\right) - \hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ijk}^{(n)} \cap E_{ik,jk}^{(n)}\right)}$$

$$= \frac{1 + O\left(W_i W_j W_k n^{-1}\right)}{1 + W_k\left[\beta\gamma + O\left(W_i W_j W_k n^{-1}\right)\right]}.$$

As the weights are independent and have finite mean, Markov's inequality can be applied to conclude that $W_iW_jW_kn^{-1}$ converges in probability to 0 as $n \to \infty$. The result then follows from bounded convergence.

(c): Applying Lemma 5.3.3 gives the bound

$$\hat{\mathbb{P}}^{(n,m)}_{F,B}\left(E^{(n)}_{ij} \mid E^{(n)}_{ik} \cap E^{(n)}_{jk}\right) \leq \frac{\hat{\mathbb{P}}^{(n,m)}_{F,B}\left(E^{(n)}_{ijk}\right) + \hat{\mathbb{P}}^{(n,m)}_{F,B}\left(E^{(n)}_{ij,ik,jk}\right)}{\hat{\mathbb{P}}^{(n,m)}_{F,B}\left(E^{(n)}_{ik,jk}\right)}$$
$$= \frac{n^{(1-\alpha)/2} + O\left(W_iW_jW_kn^{-1}\right)}{W_k\left[\beta\gamma + O\left(W_iW_jW_kn^{-1}\right)\right]}.$$

Obviously, $n^{(1-\alpha)/2} \to 0$ when $\alpha > 1$ if $n \to \infty$. Moreover, by Markov's inequality, $W_iW_jW_kn^{-1}$ converges in probability to 0 when $n \to \infty$. Hence, the fraction in (5.8) converges in probability to 0. By bounded convergence, $c^{(n,m)}_F \to 0$ when $\alpha > 1$. □

To get some intuition for Theorem 5.3.4, consider three distinct agents $i, j, k \in V^{(n)}$ for some $n \in \mathbb{N}$. Suppose that i and k share a group and that j and k share a group in a bipartite network. Then, the probability that i and j also have a group in common depends on the number of groups that the common neighbor k belongs to. Indeed, the fewer groups k belongs to, the more likely it is that i and j in fact share the same group with k. Recall that the expected number of groups that k belongs to is roughly $\beta\gamma W_kn^{(\alpha-1)/2}$. If $\alpha < 1$ (and W_k has a finite mean), the expectation of this expression goes to 0 as $n \to \infty$. In that case, it is very unlikely that k belongs to more than one group when n is large. Consequently, two edges ik and jk in the community network are likely to be generated by the same group. This implies that i and j are connected as well. Hence, when $\alpha < 1$, the clustering will be equal to 1: conditional on i and k and j and k being connected, the probability that i and j also have a link is asymptotically equal to 1. On the other hand, when $\alpha > 1$, the number of groups that k belongs to is asymptotically infinite. Hence, the fact that the agents i and j each belong to one of these groups, does not imply that it is likely that they actually belong to the same group. Hence, the clustering is asymptotically 0 when $\alpha > 1$. Finally, when $\alpha = 1$, agent k belongs to $\beta\gamma W_k$ groups on average, explaining the expression in part (b) of the theorem.

From Theorem 5.3.4 it follows that we should choose $\alpha = 1$ to obtain a random community model with nontrivial clustering. For a given distribution function F for the weights (when the weights have finite mean), the clustering can then be varied continuously between 0 and 1 by adjusting the parameters β and γ.

Furthermore, as shown in Theorem 5.3.2(b), when $\alpha = 1$, the degree distribution for a given vertex is asymptotically compound Poisson with the weight of the vertex as one of the parameters. In Appendix 5.A we show that if F is a power law distribution with a given exponent, then the degree distribution of the random community model with weight distribution F will have power law tails with the same exponent when the number of agents grows large. Since the mean of an agent's weight is normalized to 1, the asymptotic mean degree is $\beta\gamma^2$. Taken together, this means that we can obtain a random community model with arbitrary clustering and a power law degree distribution with a given exponent and a given mean by setting $\alpha = 1$ and choosing F to be a power law with the desired exponent. One can then tune the parameters β and γ to obtain the desired values for the clustering and the expected degree. This we explore in the next section.

5.4 Example: Power law weight distributions

When $\alpha = 1$, the asymptotic clustering of a random community model with weight distribution F is given by $\mathbb{E}[(1 + \beta\gamma W_k)^{-1}]$. In general, it is not possible to give an explicit expression for this expectation, but for the important case that the weight distribution F is a power law, it is possible to express the clustering in terms of some well-known complex functions. More specifically, let F be the Pareto distribution given by:

$$\forall x \geq \frac{\lambda - 2}{\lambda - 1} : \quad F(x) = 1 - \left(\frac{\lambda - 2}{\lambda - 1}\right)^{\lambda - 1} x^{-(\lambda - 1)}.$$

When $\lambda > 2$, random variables with this distribution have mean 1, as desired. The asymptotic clustering c_F is given by the integral

$$\frac{(\lambda - 2)^{\lambda - 1}}{(\lambda - 1)^{\lambda - 2}} \int_{\frac{\lambda-2}{\lambda-1}}^{\infty} (1 + \beta\gamma x)^{-1} x^{-\lambda}\, dx.$$

Defining $u := (\lambda - 2)/(x \cdot (\lambda - 1))$, we obtain

$$\begin{aligned} c_F &= \frac{1}{\beta\gamma} \frac{(\lambda - 1)^2}{(\lambda - 2)} \int_0^1 u^{\lambda - 1} \left(1 + \frac{u}{\beta\gamma}\left(\frac{\lambda - 1}{\lambda - 2}\right)\right)^{-1} du \\ &=: \frac{1}{\beta\gamma\lambda} \frac{(\lambda - 1)^2}{(\lambda - 2)}\, {}_2F_1\left(1, \lambda; 1 + \lambda; -\frac{1}{\beta\gamma}\left(\frac{\lambda - 1}{\lambda - 2}\right)\right), \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function, a function whose properties are well characterized (Abramowitz and Stegun, 1964, Ch. 15). For $\beta\gamma \geq (\lambda - 1)/(\lambda - 2)$, a

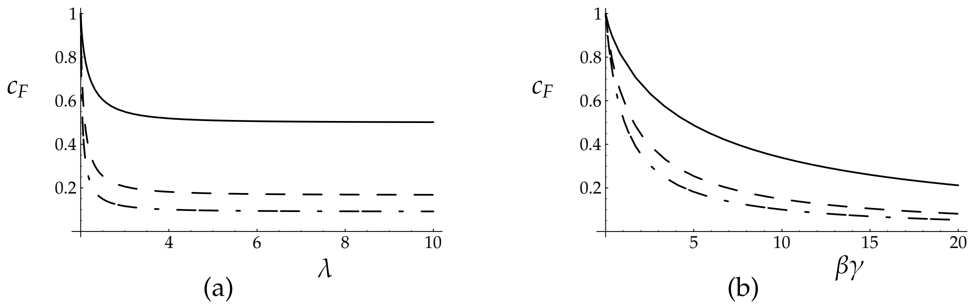


Figure 5.5. (a) The clustering as a function of λ for different values of $\beta\gamma$: $\beta\gamma = 1$ (—), $\beta\gamma = 5$ (− − −), $\beta\gamma = 10$ (− · −). (b) The clustering as a function of $\beta\gamma$ for different values of λ: $\lambda = 2.1$ (—), $\lambda = 2.5$ (− − −), $\lambda = 4$ (− · −).

series expansion of the integrand yields that

$$\begin{aligned} c_F &= \frac{1}{\beta\gamma}\frac{(\lambda-1)^2}{(\lambda-2)}\sum_{k=0}^{\infty}\left(-\frac{1}{\beta\gamma}\left(\frac{\lambda-1}{\lambda-2}\right)\right)^k\frac{1}{k+\lambda} \\ &=: \frac{1}{\beta\gamma}\frac{(\lambda-1)^2}{(\lambda-2)}\Phi\left(-\frac{1}{\beta\gamma}\left(\frac{\lambda-1}{\lambda-2}\right),1,\lambda\right), \end{aligned}$$

where Φ is the Lerch transcedent (Gradshteyn and Ryzhik, 2000). Furthermore, when λ is an integer, the expression for the clustering becomes

$$c_F = \frac{(\lambda-2)^{\lambda-1}}{(\lambda-1)^{\lambda-2}}\left[(-\beta\gamma)^{\lambda-1}\log\left(1+\frac{\lambda-1}{\beta\gamma(\lambda-2)}\right)+\sum_{\ell=1}^{\lambda-1}\frac{(-\beta\gamma)^{\lambda-1-\ell}}{\ell}\left(\frac{\lambda-1}{\lambda-2}\right)^{\ell}\right].$$

Since ${}_2F_1(a,b;c;\cdot)$ is increasing in its last argument for all $a,b,c \in \mathbb{R}$, the clustering falls monotonically in $\beta\gamma$. Also, the clustering decreases when λ increases, since more mass is then put on large values of x where the function $(1+\beta\gamma x)^{-1}$ is small. This is illustrated in Figures 5.5(a) and (b), respectively. Hence, for any $c \in (0,1)$ and a given tail exponent $\tau := \lambda - 1$, we can find a value of $\beta\gamma$ such that the clustering is equal to c. We can combine this with a condition on $\beta\gamma^2$ to fix the average degree of the random community network. Together, this determines the parameters β and γ.

5.5 Conclusions

In this chapter, we have proposed a random network model that is especially suitable to describe social and economic networks with a group structure such as R&D networks. In our model, the random community model, agents are organized in groups. The group structure determines the network structure: agents are connected in the network if and only if they share at least one group. The group structure is in turn determined by a random process: agents are assigned weights at random according to some distribution, and agents with larger weights are more likely to belong to a large number of groups. These weights can be interpreted as the "network investments" of agents. In the context of R&D networks, the groups would be the different research alliances, and the weights of firms could be their investments to become an attractive R&D partner.

We have characterized the degree distribution and the clustering of the model for different values of the parameters and for any weight distribution when the weights have finite mean. Moreover, we have shown that by choosing the parameters and the weight distribution appropriately, we can obtain a network in which both the degree distribution and the clustering can be controlled. We have illustrated this for the important example of a power law degree distribution.

There are a number of possible directions for future research. These directions can be classified into two broad categories. The first class concerns a further characterization of the current model. A first step would be to generalize the model to allow for a random number of agents. This should be a straightforward extension using the results of Bollobás et al. (2007, Sec. 8.1) for a related model. Such a generalization could be interesting for game-theoretic purposes, as the previous chapter has shown that allowing for uncertainty over the number of players in network games with incomplete information on the network structure can affect predictions.

A second direction for further research would be to characterize the correlation in the degrees of neighboring agents. Apart from the degree distribution and the clustering, an important feature of many real networks is that there is significant correlation in the degrees of neighboring nodes. That is, either vertices with a high (low) degree tend to be connected to other vertices with high (low) degree (positive correlation), or vertices with a high (low) degree tend to be connected to vertices with a low (high) degree (negative correlation). As the previous chapter has pointed out, such correlations can be important for game-theoretic predictions in network games when players have incomplete information on the network structure (also see Galeotti et al., 2006). A next step is therefore to quantify the correlations in the current model. We conjecture that the group structure will induce positive correlations in the degrees of vertices: agents who belong to a large group will have many connections, as will the agents with whom they share this group. Hence, as long as agents do not belong to too many groups, the correlation in the degrees of neighbors will be positive. This would agree well with empirical observations on social networks (Newman, 2003a). The correlation in degrees has been studied heuristically by Newman and Park (2003) for a similar model, but no exact results have been obtained. It may be hoped that the tractable form of the current model makes it possible to obtain precise results.

Also other features of the model are worth investigating. For instance, many real networks are "small worlds", meaning roughly that the distances between vertices remain small even in very large networks. It would be interesting to study the relation between the distances among vertices, the degree distribution and the clustering in the current model. The relation is not directly obvious. On the one hand, when the clustering is high, there are many "redundant" edges, connecting agents that are also indirectly connected through a common neighbor. This would mean that the average distance between agents in highly clustered networks would be larger than in networks with lower clustering with the same edge density. On the other hand, when the clustering is large, agents tend to be organized in groups, and once a path reaches a group, all members of the group are only one step away. This acts to reduce the distances in clustered networks, as it does in the original small-world model of Watts and Strogatz (1998) (see Newman and Park, 2003, for a related argument in the context of epidemics on group-structured networks).

The second class of directions for further research concerns game-theoretic applications and extensions. Firstly, one could endogenize the weights, with agents choosing their weights strategically. The distribution over agents' weights, which we have taken to be exogenous, would then be the outcome of strategic interactions. This direction is pursued by Cabrales et al. (2007) in the context of a related random network model. While in their model, players choose their networking efforts strategically to form bilateral links with other players, in the current setting, players would choose their networking efforts to join groups. This is natural in many contexts. For instance, in the context of R&D collaborations, firms can choose to invest in general purpose R&D to become an attractive R&D partner.

Secondly, an important question is how the group structure of networks affects the behavior of players located on these networks. So far, the literature on network games in which players have incomplete information about the network structure has focused on settings in which players only know their degree. However, there are settings where it does not only matter how many connections one has, but also what *kind* of connections. The sociological literature, for instance, distinguishes bridging and bonding social capital (Putnam, 2000). Bonding social capital (Coleman, 1988) refers to the value assigned to social networks between homogeneous groups, while bridging social capital (Burt, 1992) refers to that of social networks between heterogeneous groups. In the context of our model, one question one could ask is how group structures affect the functioning of risk sharing networks. In risk sharing networks the degree of individuals—the number of people with whom they share risk—is obviously an important characteristic, but it is not the only one. If a player has a high degree, it could be that he belongs to a large number of small groups, or that he belongs to a small number of large groups. If income shocks are correlated within groups but not across groups, his bridging social capital will be high in the former case. However, when players belong to a limited number of groups, clustering will be high. High clustering is often associated with low monitoring costs, as information about a player's deviant behavior can spread quickly among those with whom he interacts (e.g. Bloch et al., 2005). Hence, in the latter case, bonding social capital will be high. In our model, we can keep the average degree fixed and vary the clustering.

Another example of a setting in which group structure or clustering can have an important effect is that of information aggregation and coordination. For instance, DeMarzo et al. (2003) argue that persuasion bias, i.e., the phenomenon that individuals fail to adjust properly for repetitions in the information they obtain, is especially strong in clustered networks.[6] Similarly, Calvó-Armengol and De Martí (2007b) show that in team decision problems where players want to match their action to the state of the world as well as to other players' actions, increasing the clustering in a network (by increasing the number of edges) will increase the accuracy of agents' estimate of the state of the world and improve coordination among agents.

In the models of DeMarzo et al. (2003) and Calvó-Armengol and De Martí (2007b) (also see Calvó-Armengol and De Martí, 2007a), players do not form beliefs about their network. From social psychology it is well known that individuals believe that their networks are highly clustered (e.g. Crockett, 1982; Krackhardt and Kilduff, 1999). Therefore, it would be of interest to study how players' beliefs on

[6] Suppose two individuals discuss a certain topic after they have discussed it both with a common friend. Then, if they do not account for the fact that their discussion partner's opinion is partly based on some of the same (third party) information as their own opinion, they will double-count the third party's opinion.

the clustering of their networks affect strategic interactions in games with incomplete information on the network structure. For instance, in the setting considered by DeMarzo et al. (2003), one could study how agents' beliefs on the clustering in their network would affect their opinions. In the model of DeMarzo et al., agents act as if there is no clustering in the network. If they would believe that there is some clustering in the network, they could partly correct for their persuasion bias. This would offer a middle ground between the boundedly rational models of information spreading and opinion formation studied by DeMarzo et al. (2003) (also see Golub and Jackson, 2007) and highly rational social learning models (e.g. Gale and Kariv, 2003). The random community model we propose provides a suitable starting point, as one can vary the clustering continuously.

5.A Compound Poisson distributions with power law tails

We show that if a random variable W has a distribution function with power law tails with some exponent $\tau > 1$, then the distribution function of a sum of a Poisson(W)-distributed number of Poisson(a) random variables, $a \in (0, \infty)$, has power law tails with the same exponent.

First we need some more definitions. A random variable X has a *mixed Poisson distribution with mixing distribution* Q if

$$\forall k \in \mathbb{N}: \quad \mathbb{P}(X = k) = \mathbb{E}\left[\frac{W^k}{k!}e^{-W}\right],$$

where W is a random variable with distribution function Q. Also, define the *Gamma function* $\widetilde{\Gamma}$ by:[7]

$$\forall r \in \mathbb{C} \text{ s.t. } \mathrm{Re}[r] > 0: \quad \widetilde{\Gamma}(r) := \int_0^\infty x^{r-1}e^{-x}\,dx, \tag{5.9}$$

Let W be a random variable whose cumulative distribution function F has power law tails with exponent $\tau > 0$, i.e., $(1 - F(k)) \sim k^{-\tau}$. Consider a sequence $X_1, X_2, \ldots$ of i.i.d. random variables with a Poisson(a) distribution, where $a \in \mathbb{R}$, and let N be a random variable with a Poisson(W) distribution. Define the random

[7] For a characterization of the Gamma function, see Chapter 6 of Abramowitz and Stegun (1964).

variable Y by:

$$Y := \sum_{i=1}^{N} X_i.$$

We use the following result:

Proposition 5.A.1. *Let X be a random variable that has a mixed Poisson distribution with mixing distribution Q. If Q has density function q such that*

$$q(x) \sim x^{-\lambda}$$

for some $\lambda > 1$, then

$$\mathbb{P}(X = k) \sim k^{-\lambda}.$$

Proof. Let k be such that $k - \lambda > 0$, and for ease of notation, define

$$f_k := \mathbb{P}(X = k).$$

Then,

$$\begin{aligned} f_k &= \int_0^\infty \frac{x^k}{k!} e^{-x} q(x)\, dx \\ &\sim \frac{1}{k!} \int_{\xi_Q}^\infty x^{k-\lambda} e^{-x}\, dx \\ &= \frac{1}{k!} \left(\int_0^\infty x^{k-\lambda} e^{-x}\, dx - \int_0^{\xi_Q} x^{k-\lambda} e^{-x}\, dx \right), \end{aligned} \tag{5.10}$$

where ξ_Q is the infimum of the support of Q. Trivially,

$$\int_0^{\xi_Q} x^{k-\lambda} e^{-x}\, dx \le \left(\xi_Q\right)^{k-\lambda}.$$

To deal with the first integral in (5.10), recall the definition (5.9) of the Gamma function $\widetilde{\Gamma}$ and Stirling's formula, which states that $\widetilde{\Gamma}(r) \sim r^{r-1/2} e^{-r}$. This yields

$$\begin{aligned} \int_0^\infty x^{k-\lambda} e^{-x}\, dx &= \widetilde{\Gamma}(k - \lambda + 1) \\ &\sim (k - \lambda + 1)^{k-\lambda+1/2} e^{-(k-\lambda+1)}. \end{aligned}$$

Also note that, since $\widetilde{\Gamma}(k+1) = k!$ for $k \in \mathbb{N}$, it follows from Stirling's formula that $k! \sim (k+1)^{k+1/2}e^{-(k+1)}$. Substituting these estimates in (5.10) gives

$$f_k \sim \frac{(k-\lambda+1)^{k-\lambda+1/2}e^{-(k-\lambda+1)} - \xi_Q^{k-\lambda}}{(k+1)^{k+1/2}e^{-(k+1)}} \sim k^{-\lambda}. \qquad \square$$

Remark 5.A.2. Proposition 5.A.1 is formulated for the case that the random variable with distribution function Q is continuous, but the proof is completely analogous for the discrete case. ◁

To obtain the result that the distribution function of Y has power law tails with the same exponent τ as W, first recall that a sum of independent Poisson variables is Poisson distributed, so that Y is distributed as a Poisson(Na) random variable. Then, apply Proposition 5.A.1 to conclude that, since W has a distribution function with power law tails with exponent τ, the random variable N (which has a mixed Poisson distribution with mixing distribution F) has a distribution with power law tails with the same exponent. We can then apply Proposition 5.A.1 once more to obtain that Y, which is a Poisson(Na) random variable, has a distribution function with power law tails and exponent τ.

5.B Proofs

5.B.1 Proof of Lemma 5.3.1

Let $n \in \mathbb{N}$. For ease of notation, define

$$\begin{aligned}
K &:= \frac{\beta\gamma}{\delta} \\
A &:= \left(N_c(B_F^{(n,m)})\, n^{-(\alpha-1)/2}\right)^2 \\
B &:= n^{\alpha-1} \sum_{j\in V^{(n)}} \left(P_j^{(n)}\right)^2 .
\end{aligned}$$

Then, conditional on W_1,

$$\begin{aligned}
\mathbb{P}\left(\left(N_c(B_F^{(n,m)})\right)^2 \sum_{j\in V^{(n)}} \left(P_j^{(n)}\right)^2 > \eta\right) &= \mathbb{P}\left(AB > \eta\right) \\
&= \mathbb{P}\left(AB > \eta, B \le \frac{\eta}{K^2}\right) + \mathbb{P}\left(AB > \eta, B > \frac{\eta}{K^2}\right) \\
&\le \mathbb{P}\left(A > K^2\right) + \mathbb{P}\left(B > \frac{\eta}{K^2}\right). \qquad (5.11)
\end{aligned}$$

Consider the second term of (5.11). Using the definitions of B and $P_j^{(n)}$,

$$\begin{aligned}
B &\le \frac{\gamma^2}{n^2} \sum_{j\in V^{(n)}} W_j^2 \\
&\le \gamma^2 \left(\frac{\sum_{j\in V^{(n)}} W_j}{n}\right)\left(\frac{\max_{j\in V^{(n)}} W_j}{n}\right).
\end{aligned}$$

Since W_j has finite mean for all $j \in V^{(n)}$, by the strong law of large numbers and Lemma 2.A.2, respectively,

$$\frac{1}{n}\sum_{j\in V^{(n)}} W_j \xrightarrow{\text{a.s.}} 1, \qquad \text{and} \qquad \frac{1}{n}\max_{j\in V^{(n)}} W_j \xrightarrow{\text{P}} 0.$$

This implies that $B \xrightarrow{\text{P}} 0$, and hence

$$\mathbb{P}\left(B > \frac{\eta}{K^2}\right) \to 0.$$

Now consider the first term of (5.11):

$$\begin{aligned}
\mathbb{P}\left(A > K^2\right) &= \mathbb{P}\left(N_c(B_F^{(n,m)}) n^{-(\alpha-1)/2} > K\right) \\
&\le \frac{1}{Kn^{(\alpha-1)/2}} \mathbb{E}\left[N_c(B_F^{(n,m)})\right] \\
&= \delta W_1.
\end{aligned}$$

□

5.B.2 Proof of Lemma 5.3.3

(a): The probability that three distinct agents i, j and k in $V^{(n)}$ do not share a group in the random bipartite network (conditional on their weights) is

$$\left(1 - P_i^{(n)} P_j^{(n)} P_k^{(n)}\right)^m .$$

Using the definitions of m and $P_i^{(n)}, i \in V^{(n)}$, it follows that

$$\begin{aligned}\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ijk}^{(n)}\right) &= 1 - \left(1 - P_i^{(n)} P_j^{(n)} P_k^{(n)}\right)^m \\ &= \frac{\beta\gamma^3 W_i W_j W_k}{n^{(3+\alpha)/2}} + O\left(\frac{W_i^2 W_j^2 W_k^2}{n^{3+\alpha}}\right).\end{aligned}$$

(b): First note that the probability that there is exactly one group in the random bipartite network to which both i and j belong, conditional on their weights, is

$$mP_i^{(n)} P_j^{(n)} \left(1 - P_i^{(n)} P_j^{(n)}\right)^{m-1} = mP_i^{(n)} P_j^{(n)} + O\left(m^2 \left(P_i^{(n)}\right)^2 \left(P_j^{(n)}\right)^2\right).$$

Given that i and j share one group, the probability that i and k share exactly one of the *other* $m-1$ groups is

$$(m-1)P_i^{(n)} P_k^{(n)} \left(1 - P_i^{(n)} P_k^{(n)}\right)^{m-2} = mP_i^{(n)} P_k^{(n)} + O\left(m^2 \left(P_i^{(n)}\right)^2 \left(P_k^{(n)}\right)^2\right).$$

Finally, the conditional probability that there is a third group to which both j and k belong given that the pairs i, j and i, k share one group each is

$$1 - \left(1 - P_j^{(n)} P_k^{(n)}\right)^{m-2} = mP_j^{(n)} P_k^{(n)} + O\left(m^2 \left(P_j^{(n)}\right)^2 \left(P_k^{(n)}\right)^2\right).$$

Combining these estimates, and noting that scenarios in which i and j or i and k share more than one group have asymptotically negligible probability in comparison,

$$\begin{aligned}\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ij,ik,jk}^{(n)}\right) &= m^3 \left(P_i^{(n)}\right)^2 \left(P_j^{(n)}\right)^2 \left(P_k^{(n)}\right)^2 + \\ &\qquad O\left(m^4 \left(P_i^{(n)}\right)^2 \left(P_j^{(n)}\right)^2 \left(P_k^{(n)}\right)^2 \left(P_i^{(n)} P_j^{(n)} + P_i^{(n)} P_k^{(n)} + P_j^{(n)} P_k^{(n)}\right)\right) \\ &= \frac{\beta^3 \gamma^6 W_i^2 W_j^2 W_k^2}{n^3} + O\left(\frac{W_i^3 W_j^3 W_k^3}{n^4}\right).\end{aligned}$$

(c): The proof is analogous to the proof of part (b).

(d): The event $E_{ijk}^{(n)} \cap E_{ik,jk}^{(n)}$ occurs when there is at least one group that is shared by all three agents i, j and k and another group shared by either i and k or j and k. Denote by r the probability that agent k and at least one of the agents i and j belong to a fixed group. Then,

$$r = P_k^{(n)} \left(P_i^{(n)} + P_j^{(n)} - P_i^{(n)} P_j^{(n)}\right).$$

The probability that there is exactly one group to which all three agents i, j and k belong is

$$mP_i^{(n)}P_j^{(n)}P_k^{(n)}\left(1-P_i^{(n)}P_j^{(n)}P_k^{(n)}\right)^{m-1}=O\left(mP_i^{(n)}P_j^{(n)}P_k^{(n)}\right).$$

Conditional on the event that there is exactly one group to which i, j and k belong, the probability that there is at least one *other* group that is shared by either i and k or by j and k is

$$1-(1-r)^{m-1}=O(mr).$$

It follows that

$$\hat{\mathbb{P}}_{F,B}^{(n,m)}\left(E_{ijk}^{(n)}\cap E_{ik,jk}^{(n)}\right)=O\left(m^2P_i^{(n)}P_j^{(n)}P_k^{(n)}r\right)=O\left(\frac{W_i^2W_j^2W_k^2}{n^{(5+\alpha)/2}}\right). \qquad \square$$

Part II

Learning in Games

6 Learning with a recency bias

Summary

Often, individuals are unwilling to deviate from recent choices. This chapter, which is based on Kets and Voorneveld (2005), proposes a learning process in which players display precisely such a recency bias. It is shown that these behaviorally plausible models of adaptive play eventually settle down in so-called minimal prep sets, a set-valued solution concept for strategic games proposed by Voorneveld (2004). The current chapter thus provides a dynamic motivation for such sets.

6.1 Introduction

The behavioral economics literature provides several motivations for the common observation that agents appear somewhat unwilling to deviate from their recent choices. For instance, Tversky and Kahneman (1982, p.11) mention the bias towards recent choices as an example of the availability bias, the ease with which instances come to mind. Similarly, Schelling (1960) argues that players, when indifferent between strategies, choose the most salient strategy. In combination with the so-called recency effect (Miller and Campbell, 1959), this may explain why agents appear to have a preference for recent choices. The recency effect refers to the cognitive bias that results from disproportionate salience of recent stimuli or observations. Other motivations include models for agents displaying defaulting behavior or inertia (e.g. Vega-Redondo, 1993, 1995; Madrian, 2001), the formation of habits (Young, 1998), the use of rules of thumb (Ellison and Fudenberg, 1993), or the locking in on certain modes of behavior due to learning by doing (Grossman et al., 1977) or, as Joosten et al. (1995) express it: unlearning by not doing.

This chapter provides a class of discrete-time adjustment processes for mixed extensions of finite strategic games in which players display precisely such a bias towards recent choices. Apart from this behavioral assumption, the assumptions underlying the adaptive processes in this chapter are in conformance with much of the literature on learning (e.g. Hurkens, 1995; Fudenberg and Levine, 1998; Young, 1998): players choose best replies to beliefs that are supported by observed play in the recent past. The purpose of this chapter is to show that these behaviorally plausible models of adaptive play eventually settle down in so-called minimal prep sets, thus providing a dynamic motivation for such sets.

Minimal prep sets ("prep" is short for "preparation") were introduced and studied in a static framework in Voorneveld (2004, 2005). This set-valued solution concept for strategic games combines a standard rationality condition, stating that the set of recommended strategies to each player must contain at least one best reply to whatever belief he may have that is consistent with the recommendations to the other players, with players' aim at simplicity, which encourages them to maintain a set of strategies that is as small as possible. The latter feature discerns minimal prep sets from minimal curb sets Basu and Weibull (1991), which are product sets of pure strategies containing not just some, but *all* best responses against beliefs restricted to the recommendations to the remaining players, and from persistent retracts (Kalai and Samet, 1984), which also require the recommendations to each player to contain at least one best reply to beliefs *in a small neighborhood* of the beliefs restricted to the recommendations to the other players.

The choice of the term "preparation" in connection with minimal prep sets is motivated by the rationality requirement. Given an arbitrary belief of a player that is consistent with the recommendations to the other players, his recommended set of strategies leaves him well prepared: it contains an optimal response against all such eventualities. On the other hand, one does not have to be exhaustive to be prepared: the notion of prep sets avoids the potential avalanche effect from the requirement that all best replies against a given belief (and all best replies against all these best replies, and so on. . .) need to be included, as demanded by the curb sets of Basu and Weibull (1991).

The game in Figure 6.1 provides a simple example to illustrate the difference between pure Nash equilibria, minimal curb sets, and minimal prep sets. The game has no pure Nash equilibria. Its only—hence minimal—curb set is the entire pure strategy space $\{R_1, R_2, R_3\} \times \{C_1, C_2, C_3\}$. There are two minimal prep sets, $\{R_1, R_2\} \times \{C_1, C_2\}$ and $\{R_2, R_3\} \times \{C_2, C_3\}$, roughly speaking the "Matching pennies" subgames.

Voorneveld (2004, 2005) contains a general existence proof and a detailed comparison of minimal prep sets with Nash equilibria, rationalizability, minimal curb sets, and persistent retracts. Chapter 7 provides an axiomatic characterization of minimal prep sets and minimal curb sets. Tercieux and Voorneveld (2005) show that minimal prep sets provide sharp predictions in many economic applications,

	C_1	C_2	C_3
R_1	$1, -1$	$-1, 1$	$-100, -100$
R_2	$-1, 1$	$1, -1$	$-1, 1$
R_3	$-100, -100$	$-1, 1$	$1, -1$

Figure 6.1. A 3×3 game

including potential games, congestion games, and supermodular games, even in cases where minimal curb sets have no cutting power whatsoever and simply consist of the entire strategy space. The current chapter complements this literature by providing a dynamic motivation for minimal prep sets.

For play to settle down in a specific set, like a minimal prep or minimal curb set, players somehow need to learn to coordinate on actions from within this set. Crawford and Haller (1990, p. 577) indicate that an important coordination device is the fact that players "use asymmetric history to "label" actions that cannot be distinguished at the start". Modeling a behavioral bias, like our bias towards recent best replies, does exactly that.

The work that is closest in spirit to our analysis is that of Hurkens (1995). In both his work and in the current chapter, convergence to a set-valued solution concept is established, firstly, for discrete-time adjustment processes characterized by conditions on transition probabilities (zero or positive), secondly, for all finite games (in contrast with e.g. Young (1998), who restricts attention to weakly acyclic games), and, thirdly, for all memory lengths exceeding a certain lower bound. There are, however, important differences. The behavioral bias towards recent choices that players use to distinguish between best replies is absent in the model of Hurkens (1995): there, players indiscriminately choose best replies to their beliefs. As a consequence, players in our model need to keep track of whether one best reply was chosen more recently than another. However, this does *not* mean that a player needs to have perfect memory of his own past action choices. This is particularly clear if a player has only two actions: if both happen to be a best reply to his current belief, the action he chose in the previous round is the most recent one and therefore all he needs to recall. We return to this issue in more detail in Remarks 6.3.1 and 6.5.4.

The outline of this chapter is as follows. Definitions are recalled in Section 6.2. The evolution of play is discussed in Section 6.3. Section 6.4 contains the convergence theorem and explains the steps towards the proof. Section 6.5 discusses a more general class of adjustment processes for which play also settles down in minimal prep sets, thus providing some insight in what assumptions are essential to obtaining the convergence result. Section 6.6 contains concluding remarks. All proofs are contained in the appendices.

6.2 Preliminaries

Recall the definition of a (finite strategic) game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ (Section 2.1.1), where N is a finite, nonempty set of *players*, and each player $i \in N$ is endowed with a finite, nonempty set A_i of *actions* and a *(von Neumann Morgenstern) utility function* on the set of pure strategy profiles $A = \times_{j \in N} A_j$. Throughout this chapter, we label the players $N = \{1, \ldots, n\}$. For $i \in N$, let X_i be a nonempty subset of A_i. The set of mixed strategies of player $i \in N$ with support in X_i is denoted by $\Delta(X_i)$. Payoffs are extended to mixed strategies in the usual way. Let $i \in N$ and let $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$ be a belief of player i. The set

$$BR_i(\alpha_{-i}) = \{a_i \in A_i \mid \forall b_i \in A_i : u_i(a_i, \alpha_{-i}) \geq u_i(b_i, \alpha_{-i})\}.$$

is the set of pure best replies of player i against α_{-i}.

Definition 6.2.1. *A* prep set *of a game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ is a nonempty product set $Q = \times_{i \in N} Q_i \subseteq A$ such that for each $i \in N$ and each belief $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(Q_j)$ of player i, the set Q_i contains* at least one *best response of player i against his belief:*

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(Q_j) : \quad BR_i(\alpha_{-i}) \cap Q_i \neq \emptyset.$$

A prep set Q is minimal *if no prep set is a proper subset of Q.*

In the adaptive processes we study, a game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is played once every period in discrete time. A *history (of play)* is a sequence $h = (a^1, \ldots, a^L) \in A^L$ of some arbitrary length $L \in \mathbb{N}$, whose leftmost element

$$\ell(h) := a^1 \in A$$

is interpreted as the action profile chosen in the previous period according to history h, with $\ell_i(h) := a_i^1 \in A_i$ the action played by $i \in N$. Generally, the k-th element from the left is the action profile $a^k \in A$ chosen $k = 1, \ldots, L$ periods ago.

A *successor* of history $h = (a^1, \ldots, a^L)$ is a history obtained after one more period of play, i.e., a history $h' = (b^1, b^2, \ldots, b^{L+1})$ obtained from h by appending a new leftmost element: $b^1 \in A$ and $b^k = a^{k-1}$ for all $k = 2, \ldots, L + 1$.

Fix a history $h = (a^1, \dots, a^L)$ and a player $i \in N$. The set of actions chosen by i during the last[1] $k \in \{1, \dots, L\}$ rounds of history h is denoted by

$$\lambda_i(h,k) := \{a_i^1, \dots, a_i^k\}.$$

The *order* $o_{i,h}$ of player i's actions in history h is defined as follows: his most recent action, i.e., the first encountered action is $o_{i,h}(1) := a_i^1$ and, inductively, for $k = 2, \dots, |\{a_i^1, \dots, a_i^L\}|$, the k-th encountered action is $o_{i,h}(k) := a_i^m$ with

$$m = \min\{q \in \{1, \dots, L\} \mid a_i^q \notin \{o_{i,h}(1), \dots, o_{i,h}(k-1)\}\}.$$

Example 6.2.2. Consider a two-player game with $N = \{1,2\}$ and action spaces $A_1 = \{T, B\}, A_2 = \{L, R\}$. Consider the history

$$h = ((T,R),(B,R),(B,L))$$

of length three. Then $\ell(h) = (T,R)$. The set of actions player 1 chose during the most recent two periods is $\lambda_1(h,2) = \{T,B\}$, whereas $\lambda_2(h,2) = \{R\}$. As to orders, player 1's action T is encountered first, then B, so $o_{1,h}(1) = T, o_{1,h}(2) = B$. Similarly, $o_{2,h}(1) = R, o_{2,h}(2) = L$. ◁

6.3 Adaptive play

This section presents a class of Markov chains to model adaptive play with a bias towards choices from the recent past. A game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is played once every period in discrete time. In line with much of the literature on learning models (e.g. Hurkens, 1995; Fudenberg and Levine, 1998; Young, 1998), players choose, at each moment in time, best replies to beliefs supported by a limited horizon of observed past play of fixed length $T \in \mathbb{N}$.[2] Consequently, the *state space* H is defined to consist of all histories $h = (a^1, \dots, a^L)$ with length at least T, i.e., $h \in \cup_{K \in \mathbb{N}, K \geq T} A^K$.

Having defined the set H of states, we proceed to *transition probability functions* $P : H \times H \to [0,1]$, where $P(h,h')$ is the probability of moving from state $h \in H$ to state $h' \in H$ in one period and $\sum_{h' \in H} P(h,h') = 1$ for all $h \in H$. To do so, beliefs and responses to them need to be modeled.

[1] Hence our choice of the alliterative λ (lambda).

[2] Our adjustment processes are defined for a fixed game G and memory length T; to simplify notation, indices G and T are suppressed.

Beliefs: Players' beliefs are based on observed play in the past $T \in \mathbb{N}$ periods. Formally, for each state $h \in H$, if the sequence of action profiles played in the past T periods is $(a^1, \dots, a^T) \in A^T$, then player i's beliefs are drawn from a probability measure $\mathbb{P}_{(i,(a^1,\dots,a^T))}$ over the set of beliefs (with its standard topology and Borel σ-algebra)

$$\times_{j \in N \setminus \{i\}} \Delta(\{a_j^1, \dots, a_j^T\}) = \times_{j \in N \setminus \{i\}} \Delta(\lambda_j(h, T))$$

with support in the product set of actions chosen in the previous T periods. For the convergence result, the exact probabilities are irrelevant: what matters is that some are positive, others zero. We therefore refrain from restricting attention to specific belief formation processes or updating procedures. As long as beliefs are sufficiently diverse—see Remark 6.3.2 or the related discussion in Hurkens (1995, pp. 310–311)—it is immaterial how they are formed.

Responses: Given a belief $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(\lambda_j(h, T))$, it is assumed that player i chooses the most recent best reply to α_{-i} if such a best reply exists, that is, if in state h some best reply to α_{-i} has been played before. Otherwise, player i chooses each best reply to α_{-i} with positive probability, i.e., it is drawn from a probability measure $\mathbb{P}_{\alpha_{-i}}$ over A_i whose support coincides with the set of best replies $BR_i(\alpha_{-i})$. Players thus have a bias towards recent choices.[3]

Together, the probability distributions $\mathbb{P}_{(i,(a^1,\dots,a^T))}$ that fix for each player $i \in N$ and account of recent play $(a^1, \dots, a^T) \in A^T$ the way beliefs are drawn, and the assumption that players are biased towards recent choices, determine the transition probabilities $P(h, h') \in [0, 1]$ for each pair of states $(h, h') \in H \times H$. If $P(h, h') > 0$, then histories $h, h' \in H$ satisfy the following two conditions:

P1: h' is a successor of $h := (a^1, \dots, a^L)$;

P2: For each $i \in N$, $\ell_i(h')$ is a best reply to some belief $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(\lambda_j(h, T))$. It is the most recent best reply, if such a best reply exists. Formally:
- $\ell_i(h') \in BR_i(\alpha_{-i})$;
- if $BR_i(\alpha_{-i}) \cap \{a_1^1, \dots, a_i^L\} \neq \emptyset$, then $\ell_i(h') = a_i^k$, where

$$k = \min\{m \in \{1, \dots, L\} \mid BR_i(\alpha_{-i}) \cap \{a_i^1, \dots, a_i^m\} \neq \emptyset\}.$$

[3] The probability of choosing $a_i \in A_i$ against beliefs to which no best reply was chosen before is

$$\int_{\{\alpha_{-i} | BR_i(\alpha_{-i}) \cap \lambda_i(h,L) = \emptyset\}} \mathbb{P}_{\alpha_{-i}}(a_i) \, \mathrm{d}\mathbb{P}_{(i,(a^1,\dots,a^T))},$$

i.e., $\alpha_{-i} \mapsto \mathbb{P}_{\alpha_{-i}}(a_i)$ is assumed to be Borel measurable.

Condition P1 is standard for discrete-time processes, stating that between time periods the game is played once: the process moves from a history h to one of its successors h'. Condition P2 requires that for each $i \in N$, $\ell_i(h')$ is a best reply to some belief $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(\lambda_j(h, T))$, and that it is the most recent best reply, if such a best reply exists. This condition thus states, firstly, that the process P is a best-reply process: the action $\ell_i(h') \in A_i$ chosen by each player $i \in N$ is a best reply to some belief α_{-i} about the remaining players' behavior based on recent experience, i.e., with support in $\times_{j \in N \setminus \{i\}} \Delta(\lambda_j(h, T))$. Secondly, it models the bias towards recent choices: whenever possible, each player $i \in N$ chooses the most recent best reply to belief α_{-i}.

Let $\mathscr{P}$ be the class of transition probability functions P achieved in this way, i.e., from probability distributions $\{\mathbb{P}_{(i,(a^1,\ldots,a^T))} : i \in N, (a^1, \ldots, a^T) \in A^T\}$ and the behavioral bias, and with $P(h, h') > 0$ if and only if states $h, h' \in H$ satisfy conditions P1 and P2.

Remark 6.3.1. The behavioral bias towards recent choices modeled in P2 requires that a player with multiple best replies against his current belief recalls whether one of them was played more recently than another. However, this does *not* require players to have perfect memory about their own actions: if you played one best reply yesterday and another a week ago, your choice is independent of whether you also adopted these actions further away in the past. All that matters is that each player $i \in N$ in history $h \in H$ recalls the order $o_{i,h}$ defined in Section 6.2. This is a considerably more modest requirement than remembering the entire history of own actions: $o_{i,h}$ specifies a simple linear order of at most $|A_i|$ actions. Between consecutive rounds of play, this linear order either remains the same or changes in the following way: the action ranked first (the most recent action) is changed and the other actions are moved one step down the ladder. For instance, even after numerous rounds of play, the only thing a player with just two actions needs to recall from his own past is last period's action. $\triangleleft$

Remark 6.3.2. Inherent in the definition of the class $\mathscr{P}$ of transition probability functions is that beliefs must be "sufficiently diverse" to assure that player $i \in N$ has a positive probability of selecting $a_i \in A_i$ whenever it is a (most recent) best reply to some belief over recent past play. More specifically, by P2, player i is tempted to play a_i against beliefs α_{-i} over recent past play to which it is the most recent best reply or—if no such most recent best reply exists—to which it is an arbitrary best reply. If the set of such "tempting" beliefs is nonempty, player i assigns positive probability to it. $\triangleleft$

For each $k \in \mathbb{N}$, let $P^k : H \times H \to [0,1]$ denote the k-step transition probabilities of our Markov process with transition probability function $P \in \mathscr{P}$: $P^1 = P$ and $P^k = P \circ P^{k-1}$ for all $k > 1$.

6.4 Convergence and steps towards the proof

This section presents the main result of this chapter. Theorem 6.4.1 states, for each game G and adjustment process in the class $\mathscr{P}$, that if beliefs are based on recent experience of sufficient length T, then play will eventually settle down in a minimal prep set. The steps of the proof are briefly explained in this section; the proof itself is contained in Appendix 6.A. Proposition 8.5.1 in Chapter 8 presents a short proof of our convergence result for the special case of minority games and may therefore provide the reader with helpful intuition for the general case.

Given a game G and an adjustment process $P \in \mathscr{P}$, the process is said to *eventually settle down* in a minimal prep set of G if the probability that the process after k steps is in a state $h \in H$ where

- the most recently played action profile $\ell(h)$ lies in some minimal prep set Q of G:

  $$\ell(h) \in Q$$

- all future action profiles remain inside Q:

  $$\ell(h') \in Q \text{ whenever } P^m(h,h') > 0 \text{ for some } m \in \mathbb{N}, h' \in H,$$

converges to one as k goes to infinity.

Theorem 6.4.1. *Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game. Let the horizon $T \in \mathbb{N}$ of recent past play on which beliefs are based satisfy*

$$T \geq \max \left\{ \sum_{i \in N} |A_i| - n + 1, 2|A_1|, \ldots, 2|A_n| \right\}. \tag{6.1}$$

If $P \in \mathscr{P}$, then play eventually settles down in a minimal prep set of G.

Remark 6.4.2. If the game has several minimal prep sets, the one selected by the learning process typically depends on initial conditions. For instance, if the initial state is such that the collection of most recent actions is a minimal prep set Q, the process settles down in Q. ◁

Remark 6.4.3. Condition P2 assures that play will not settle down in proper subsets of a minimal prep set. To see this, suppose play settles down in a product set Y properly contained in a minimal prep set Q. Since Y is not a prep set, there is a player i with a belief over recent past play against which Y_i contains no best reply. Condition P2 assures that player i with positive probability chooses such a best reply, i.e., an action outside Y_i, contradicting the assumption that play has settled down in Y. A similar intuition is used in the proof of the Theorem in Appendix 6.A (Lemma 6.A.1). ◁

Remark 6.4.4. The statement of Theorem 6.4.1 follows the traditional pattern (cf. Hurkens, 1995; Fudenberg and Levine, 1998; Young, 1998): if memory is 'sufficiently long', play settles down in sets of a certain type. Thus, we have indicated a sufficient length in (6.1), without aiming at sharpness. When one exploits specific features of a game, the bound can sometimes be relaxed. The analysis in Section 8.5 provides an instance of this. ◁

Steps towards the proof: The proof of Theorem 6.4.1 proceeds in four steps:

Step 1: Let $h_0 \in H$. The process moves with positive probability in $T-1$ steps to a state $h_1 \in H$ where the product set $\times_{i \in N} \lambda_i(h_1, T) \subseteq A$ of actions played in the past T periods is a prep set.

The intuition behind this step is the following. If, for some state $g \in H$ and some $k \leq T$, the product set $\times_{i \in N} \lambda_i(g, k)$ is a prep set, then with positive probability, players choose actions from this prep set for $T-k$ periods in a row. If on the other hand, $\times_{i \in N} \lambda_i(g, k)$ is not a prep set, then there is a nonempty set of players $i \in N$ with a belief $\alpha^*_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(\lambda_j(g, k))$ over play in the past k periods to which $\lambda_i(g, k)$ does not contain a best reply. In that case, one can construct a sequence of states $g_1, g_2, \ldots \in H$ with $g_1 = g$, $P(g_k, g_{k+1}) > 0$ for all $k = 1, 2, \ldots$, such that the sequence of product sets $\times_{i \in N} \lambda_i(g_k, k)$ is strictly increasing with respect to set inclusion (see Lemma 6.A.1 in Appendix 6.A). All these sets are contained in the finite set A of action profiles which is a prep set. Since there are only finitely many actions, the sequence reaches, after a finite number of steps, a state $g_K \in H$ where $\times_{i \in N} \lambda_i(g_K, K)$ is a prep set. From that state onwards, players choose with positive probability actions from the prep set for $T-K$ periods in a row.

Step 2: From state h_1, the process moves with positive probability in a finite number of steps to a state $h_2 \in H$ where $Q := \times_{i \in N} \lambda_i(h_2, T)$ is a minimal prep set.

Indeed, let $Q = \times_{i \in N} Q_i \subseteq \times_{i \in N} \lambda_i(h_1, T)$ be a minimal prep set. The proof of this step relies on the fact that one can—under some conditions—perform so-called neighbor switches: from a state $h \in H$, the process moves with positive probability in T steps to a state $h' \in H$ whose horizon of recent past play is identical to the one in h, except that two neighboring actions of some player have changed places (see Lemma 6.A.6). As all permutations of a finite set can be obtained by a chain of such neighbor switches, the process moves with positive probability from state h_1 to a state h' where, for each player $i \in N$, $\lambda_i(h', |Q_i|) = Q_i$, i.e., the $|Q_i|$ most recent actions of each player i are exactly those in his component of the minimal prep set Q. Then it is easy to show that the process moves with positive probability to a state h_2 within a finite number of steps such that $\times_{i\in N}\lambda_i(h_2, T) = Q$ is a minimal prep set.

Step 3: After reaching state h_2, all action profiles that are played with positive probability lie in Q, i.e.,

$$\forall k \in \mathbb{N}, \forall h \in H : P^k(h_2, h) > 0 \Rightarrow \ell(h) \in Q.$$

In state h_2, the product set $\times_{i\in N}\lambda_i(h_2, T) = Q$ is a minimal prep set, which by definition contains at least one best reply to whatever belief a player may have about other players' choices from Q. Hence, by induction, the actions from minimal prep set Q will always be fresher in players' recollection of past play than actions outside Q, so that to any belief that a player i may have about opponents' play, there is an action in Q_i that is the most recent best reply. Hence, from state h_2 onwards, each player $i \in N$ only chooses actions from Q_i.

Step 4: Starting from an arbitrary history h_0, Step 1 and 2 show that there is a positive probability of proceeding to a history h_2 in a finite number of steps, after which play settles down in a minimal prep set, i.e., a positive probability of proceeding to an absorbing set of states in finitely many steps. Since the initial history was chosen arbitrarily, this eventually happens with probability one, finishing the proof.

6.5 Allowing for other behavioral biases

This section describes a more general class of adjustment processes, permitting other behavioral biases, for which play still converges to a minimal prep set of the game. Example 6.5.2 provides an explicit scenario where players no longer strictly focus on the most recent best replies, but choose more freely among their best replies. In addition to enlarging the class of processes that settle down in a minimal prep set, this section also gives some insight into which assumptions on learning processes are essential to obtain convergence to minimal prep sets.

To show that processes from $\mathscr{P}$ eventually settle down in minimal prep sets, the proof of Steps 1 and 2 of Theorem 6.4.1 (see Appendix 6.A) uses that certain transition probabilities are positive to show that the process can move from any initial state $h_0 \in H$ in a finite number of steps to a state $h_2 \in H$ where $\times_{i\in N}\lambda_i(h_2, T)$ is a minimal prep set. The proof of Step 3 uses that certain transition probabilities are zero to show that each player—once such a state h_2 is reached—continues to play action profiles from the minimal prep set. These conditions on the transition probabilities are motivated by assuming that players, whenever possible, choose the most recent best reply to a certain belief. However, any class of adjustment processes that respects these conditions on the sign of the transition probabilities will converge to minimal prep sets. Hence, one can easily extend the class of adjustment processes with this limit behavior.

In particular, suppose that for each player $i \in N$, the response to a belief drawn from recent past play in state $h \in H$ is chosen according to a probability distribution (mixed strategy) $R_{i,h} \in \Delta(A_i)$ depending on (*i*) the account $(a^1, \ldots, a^T)$ of recent past play, and (*ii*) the order in which the players' used actions appear in h. That is, for each pair of states $h = (a^1, \ldots, a^L), g = (b^1, \ldots, b^K) \in H$:

$$\left.\begin{array}{l}(a^1, \ldots, a^T) = (b^1, \ldots, b^T) \\ o_{i,h} = o_{i,g} \text{ for all } i \in N\end{array}\right\} \Rightarrow R_{i,h} = R_{i,g} \text{ for all } i \in N. \tag{6.2}$$

The collection of functions $R = (R_{i,h})_{i\in N, h\in H}$ determines, for each pair of states $h, h' \in H$, the transition probability $P_R(h, h') \in [0, 1]$. If $P_R(h, h') > 0$, then h' is a successor of h (condition P1) and

$$P_R(h, h') = \prod_{i\in N} R_{i,h}(\ell_i(h'))$$

is the probability of the players choosing action profile $\ell(h')$. Let $\widetilde{\mathscr{P}}$ be the collection of transition probability functions $\{P_R : H \times H \to [0, 1] \mid R = (R_{i,h})_{i\in N, h\in H}\}$ satisfying the restrictions on the sign of the transition probabilities instrumental to the proof of Theorem 6.4.1, i.e., for each pair of histories $h, h' \in H$:

(α) If P1 and P2 hold, then $P_R(h, h') > 0$.

(β) If the product set of actions played during the most recent $k \geq T$ rounds of h is a minimal prep set, play settles down within this set. Formally, if $Q := \times_{i\in N}\lambda_i(h, k)$ is a minimal prep set for some $k \geq T$ and $P_R(h, h') > 0$, then $\times_{i\in N}\lambda_i(h', k + 1) = Q$, i.e., $\ell(h') \in Q$.

Let us start by verifying that $\mathscr{P} \subseteq \widetilde{\mathscr{P}}$.

Remark 6.5.1. Let $P \in \mathscr{P}$. The probability $R_{i,h}(a_i)$ that player $i \in N$ in state $h = (a^1, \ldots, a^L) \in H$ chooses action $a_i \in A_i$ equals the probability of drawing a belief α_{-i} from $\mathbb{P}_{(i,(a^1,\ldots,a^T))}$ to which:

(*i*) a_i is the most recent best reply, or, alternatively,

(*ii*) no best reply was played before, but response a_i is drawn from $\mathbb{P}_{\alpha_{-i}}$.

Hence, there are functions $R = (R_{i,h})_{i \in N, h \in H}$ such that $P = P_R$. Conditions (α) and (β) follow trivially from P1 and P2 in the definition of $\mathscr{P}$. Conclude that $P \in \widetilde{\mathscr{P}}$. ◃

The set inclusion $\mathscr{P} \subseteq \widetilde{\mathscr{P}}$ is strict: one easily finds processes in $\widetilde{\mathscr{P}} \setminus \mathscr{P}$ by letting players choose more freely among recent best replies.

Example 6.5.2. Let $h = (a^1, \ldots, a^L) \in H$, $i \in N$, and let $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(\lambda_j(h,T))$ be a belief over recent past play. If $BR_i(\alpha_{-i}) \cap \{a_i^1, \ldots, a_i^L\} \neq \emptyset$, let $Y_i(h, \alpha_{-i}) \subseteq A_i$ be the singleton set consisting of the most recent best reply to α_{-i}; otherwise, let $Y_i(h, \alpha_{-i}) = BR_i(\alpha_{-i})$ consist of all best replies. Take the union over all beliefs over recent past play to obtain

$$Y_i(h) = \cup_{\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(\lambda_j(h,T))} Y_i(h, \alpha_{-i}).$$

For condition (α) to hold, $R_{i,h}$ must assign positive probability to each action in $Y_i(h)$. But player i can choose more freely among recent best replies, not just the *most* recent ones. Let

$$Z_i(h) = BR_i(\times_{j \in N \setminus \{i\}} \Delta(\lambda_j(h,T))) \cap \lambda_i(h,T)$$

be the set of all of i's best replies to beliefs over $\times_{j \in N \setminus \{i\}} \lambda_j(h,T)$ that he played during the horizon of recent past play T. Fix a probability distribution $R_{i,h}$ over A_i with support $Y_i(h) \cup Z_i(h)$. For the purpose of illustration, take a simple uniform distribution:

$$\forall a_i \in A_i: \quad R_{i,h}(a_i) = \begin{cases} 1/|Y_i(h) \cup Z_i(h)|, & \text{if } a_i \in Y_i(h) \cup Z_i(h); \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

Alternatively, one could for instance assign higher probability to more recent best replies in $Y_i(h)$ than to less recent best replies in $Z_i(h) \setminus Y_i(h)$. With $R = (R_{i,h})_{i \in N, h \in H}$ as in (6.3), it follows easily that $P_R \in \widetilde{\mathscr{P}}$:

- condition (6.2) holds: if states $h, g \in H$ satisfy the conditions in (6.2), then $Y_i(h) = Y_i(g)$ and $Z_i(h) = Z_i(g)$;

- condition (α) holds: each player $i \in N$ assigns positive probability to all actions in $Y_i(h)$;
- condition (β) holds: if $Q := \times_{i\in N}\lambda_i(h,k)$ is a minimal prep set for some $k \geq T$, then $Y_i(h) \subseteq Q_i$ and $Z_i(h) \subseteq \lambda_i(h,T) \subseteq Q_i$ for all $i \in N$. Hence, using (6.3), $\ell_i(h') \in Y_i(h) \cup Z_i(h) \subseteq Q_i$ for all $i \in N$, i.e., $\ell(h') \in Q$.

Since the process also assigns positive probability to possible other recent best replies over observed past play during the last T rounds, $P_R \notin \mathscr{P}$. ◃

By construction, if players' memory is sufficiently long, processes in $\widetilde{\mathscr{P}}$ eventually settle down in minimal prep sets. To summarize (the proof can be found in Appendix 6.B):

Proposition 6.5.3. *Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a game and let $T \in \mathbb{N}$. Then $\mathscr{P} \subset \widetilde{\mathscr{P}}$. Moreover, if $P_R \in \widetilde{\mathscr{P}}$ and the horizon $T \in \mathbb{N}$ of recent past play is sufficiently long, then play eventually settles down in a minimal prep set of G.*

Finally, we show why it is essential for convergence to minimal prep sets that players keep track of the order $o_{i,h}$ defined in Section 6.2, rather than just the order of their actions over the past T rounds of play.

Remark 6.5.4. If players remember from their own past only their actions in the previous T periods, the resulting processes need not converge to minimal prep sets: none of the players $i \in N$ can condition his behavior on the order $o_{i,h}$ of actions chosen more than T periods ago at state h. To see why this prevents convergence to minimal prep sets, refer back to Figure 6.1. Suppose that over the past T rounds, players have chosen the actions from minimal prep set $X = \{R_1, R_2\} \times \{C_1, C_2\}$. Why would play not settle down in this product set of actions? Suppose players play (R_1, C_2) at a given round, which are best replies to beliefs over X. In response to these actions, there is a positive probability that they choose (R_2, C_2) for T consecutive periods. At that point, player 2's only feasible belief over past play is that player 1 chooses R_2. Player 2 recalls only his past T actions, i.e., just C_2 which is not a best reply to R_2. Therefore, he chooses among the best replies $\{C_1, C_3\}$ to R_2, which means that he may "jump" outside the minimal prep set X by selecting C_3. ◃

6.6 Conclusions

The purpose of this chapter was to study discrete-time best-response processes with a behaviorally plausible bias towards recent actions. Such processes were shown to settle down in minimal prep sets. This dynamic motivation complements other work on minimal prep sets in a static environment, where the concept is compared with many other solution concepts (Voorneveld, 2004, 2005, and Chapter 7 of this thesis) and shown to have genuine "bite" in economic applications (Tercieux and Voorneveld, 2005), even in cases where, for instance, the minimal curb sets of Basu and Weibull (1991) have no cutting power whatsoever.

Several modifications of these processes were discussed in the previous section. It is impossible to do justice to the long list of choice biases discussed in the behavioral economics literature. An interesting direction for future research would be to study which assumptions underlying different learning processes determine which solution concept these processes converge to. A first step in this direction is taken in Section 6.5.

6.A Proof of Theorem 6.4.1

Fix a game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$, length $T \in \mathbb{N}$ of recent past play with $T \geq \max\{\sum_{i\in N} |A_i| - n + 1, 2|A_1|, \ldots, 2|A_n|\}$, and an adjustment process with transition probability function $P \in \mathscr{P}$. First, we need some additional notation. Fix an arbitrary history $h = (a^1, \ldots, a^L) \in H$ and player $i \in N$. The action player i chose in h a number of $t \in \{1, \ldots, T\}$ periods ago is denoted by

$$a_i(h,t) := a_i^t$$

and the action player i chose in h exactly T periods ago is denoted by

$$\tau_i(h) := a_i^T = a_i(h,T).$$

Action $a_i \in \lambda_i(h,T)$ is *blocked in h* if there is no belief $\alpha_{-i} \in \times_{j\in N\setminus\{i\}}\Delta(\lambda_j(h,T))$ against which it is the most recent best reply. Finally, the *frequency* with which player i chose action $a_i \in \lambda_i(h,T)$ during the past T rounds of history h is

$$f_i(h,a_i) = \left|\left\{t \in \{1,\ldots,T\} \;\middle|\; a_i(h,t) = a_i\right\}\right|.$$

We now prove the four steps of Theorem 6.4.1.

6.A.1 Proof of Step 1

Step 1: Let $h_0 \in H$. The process moves with positive probability in $T-1$ steps to a state $h_1 \in H$ where the product set $\times_{i \in N} \lambda_i(h_1, T) \subseteq A$ of actions played in the past T periods is a prep set. The proof uses the following lemma.

Lemma 6.A.1. *Consider state $h = (a^1, \dots, a^L) \in H$ and a number $t \in \{1, \dots, T-1\}$.*

(a) Suppose that $\times_{i \in N} \lambda_i(h, t) \subseteq A$ is not a prep set. Then the process moves with positive probability to a successor h' of h where

$$\times_{i \in N} \lambda_i(h, t) \subset \times_{i \in N} \lambda_i(h', t+1). \tag{6.4}$$

(b) Suppose that $\times_{i \in N} \lambda_i(h, t) \subseteq A$ is a prep set. Then the process moves with positive probability to a successor h' of h where

$$\times_{i \in N} \lambda_i(h, t) = \times_{i \in N} \lambda_i(h', t+1). \tag{6.5}$$

Proof. (a): Since $\times_{i \in N} \lambda_i(h, t) \subseteq A$ is not a prep set, there is a nonempty set $S \subseteq N$ of players such that each players $i \in S$ has a belief $\alpha^*_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(\lambda_j(h, t))$ over the play in the past t periods to which $\lambda_i(h, t)$ does not contain a best reply: $BR_i(\alpha^*_{-i}) \cap \lambda_i(h, t) = \emptyset$. Fix such a belief α^*_{-i} for each $i \in S$ and let $b_i \in BR_i(\alpha^*_{-i})$ be a best reply to α^*_{-i} chosen in accordance with P2: it is the most recent one if $BR_i(\alpha^*_{-i}) \cap \{a_i^1, \dots, a_i^L\} \neq \emptyset$. For each $i \in N \setminus S$, let $b_i \in \lambda_i(h, t)$ be the most recent best reply to an arbitrary belief over play in the past t periods. Such a best reply exists by definition of S. By P1 and P2, the process moves with positive probability from state h to successor $h' = (b, a^1, \dots, a^L)$. Now (6.4) holds by construction: if $i \in N \setminus S$, then $b_i \in \lambda_i(h, t)$, so $\lambda_i(h, t) = \lambda_i(h', t+1)$, and if $i \in S$, then $b_i \notin \lambda_i(h, t)$, so $\lambda_i(h, t) \subset \lambda_i(h, t) \cup \{b_i\} = \lambda_i(h', t+1)$.

(b): Fix, for each $i \in N$, a belief $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(\lambda_j(h, t))$ over the play in the past t periods. Since $\times_{i \in N} \lambda_i(h, t)$ is a prep set, there is an action $b_i \in \lambda_i(h, t)$ which is the most recent best reply to this belief. By P1 and P2, the process moves with positive probability from h to $h' = (b, a^1, \dots, a^L)$. Since $b_i \in \lambda_i(h, t)$ for all $i \in N$, it follows that $\lambda_i(h', t+1) = \lambda_i(h, t)$, so (6.5) holds. □

Applying Lemma 6.A.1 $T-1$ times, one can construct a sequence $g_1, \dots, g_T$ in H with $g_1 := h_0$ and for all $k = 1, \dots, T-1$: $P(g_k, g_{k+1}) > 0$ and

$$\times_{i \in N} \lambda_i(g_k, k) \subseteq \times_{i \in N} \lambda_i(g_{k+1}, k+1),$$

with strict inclusion if $\times_{i \in N} \lambda_i(g_k, k)$ is not a prep set and equality otherwise. The sequence of product sets $\times_{i \in N} \lambda_i(g_k, k)$ in A can increase strictly during at most

$\sum_{i\in N} |A_i| - n$ steps: the action space A is a prep set containing $\sum_{i\in N} |A_i|$ actions; $\times_{i\in N}\lambda_i(g_1, 1)$ captures n of them, and in each step at least one action is added until a prep set is reached. Hence, the sequence has to reach, after $K \leq \sum_{i\in N} |A_i| - n$ steps, a state $g_{K+1} \in H$ where $\times_{i\in N}\lambda_i(g_{K+1}, K+1)$ is a prep set.[4] In the final $T-K-1$ steps, we proceed to a state g_T, where

$$\times_{i\in N}\lambda_i(g_T, T) = \times_{i\in N}\lambda_i(g_{T-1}, T-1) = \cdots = \times_{i\in N}\lambda_i(g_{K+1}, K+1)$$

remains a prep set. Taking $h_1 := g_T$ finishes the proof of Step 1.

6.A.2 States without blocked actions

In this section it is shown that the process moves with positive probability within a finite number of steps from a state $h \in H$ such that $\times_{i\in N}\lambda_i(h, T)$ is a prep set to a state $h' \in H$ where $\times_{i\in N}\lambda_i(h', T) \subseteq \times_{i\in N}\lambda_i(h, T)$ is a prep set without blocked actions. This is established in Lemma 6.A.3, which uses Lemma 6.A.2. Furthermore, in Lemma 6.A.4 it is shown that when considering a sequence $g_1, \ldots, g_K$ of states such that, for all $k = 1, \ldots, K$, $\times_{i\in N}\lambda_i(g_k, T)$ is a prep set and $\times_{i\in N}\lambda_i(g_1, T) \supseteq \cdots \supseteq \times_{i\in N}\lambda_i(g_K, T)$, we can assume without loss of generality that none of the states $g_1, \ldots, g_K$ contains a blocked action. This result is used in the lemmata of the following subsections.

Lemma 6.A.2. *Let $h \in H$ be such that $\times_{i\in N}\lambda_i(h, T)$ is a prep set. For each player $i \in N$, define $\beta_i(h) \in \lambda_i(h, T)$ as follows:*

- *if $\tau_i(h)$ is blocked, let $\beta_i(h) \in \lambda_i(h, T)$ be an arbitrary non-blocked action;*
- *if $\tau_i(h)$ is not blocked, let $\beta_i(h) = \tau_i(h)$.*

Set $h' = (\beta(h); h)$, with $\beta(h) = (\beta_i(h))_{i\in N}$. Then,

(i) $P(h, h') > 0$;
(ii) $\times_{i\in N}\lambda_i(h', T) \subseteq \times_{i\in N}\lambda_i(h, T)$;
(iii) $\times_{i\in N}\lambda_i(h', T)$ *is a prep set.*

Proof. For all $i \in N$, $\beta_i(h) \in \lambda_i(h, T)$ is not blocked by definition: there is a belief $\alpha_{-i} \in \times_{j\in N\setminus\{i\}}\Delta(\lambda_j(h, T))$ against which $\beta_i(h)$ is the most recent best reply. By P1 and P2, (*i*) holds. Since $\beta_i(h) \in \lambda_i(h, T)$ for all $i \in N$, (*ii*) holds. To prove (*iii*), let $i \in N$ and $\alpha_{-i} \in \times_{j\in N\setminus\{i\}}\Delta(\lambda_j(h', T))$. To show: $BR_i(\alpha_{-i}) \cap \lambda_i(h', T) \neq \emptyset$. By construction, $\lambda_i(h', T)$

[4] This motivates the term $M := \sum_{i\in N} |A_i| - n + 1$ in the lower bound on T in (6.1): reaching a prep set can take $M-1$ steps; recalling the added actions and those in g_1 can consequently take a memory length M.

equals either $\lambda_i(h,T)$ or, if $\tau_i(h)$ was blocked and chosen only once in the most recent T periods of history h, $\lambda_i(h,T) \setminus \{\tau_i(h)\}$. Consequently, $\lambda_i(h',T)$ still contains a best reply to every belief over $\times_{j\in N\setminus\{i\}}\Delta(\lambda_j(h,T))$, in particular to every belief over the subset $\times_{j\in N\setminus\{i\}}\Delta(\lambda_j(h',T))$. □

Claim (*ii*) means that in going from h to $h' = (\beta(h);h)$, the pool of feasible beliefs is weakly decreased. This implies that if $a_i := \tau_i(h)$ was blocked in h, but was chosen more than once in the last T rounds of h, i.e., if $a_i \in \lambda_i(h',T)$, then it remains blocked:

(*iv*) if $a_i := \tau_i(h)$ was blocked in h and $a_i \in \lambda_i(h',T)$, then it is blocked in h'.

By definition, blocked actions are not chosen in going from h to h'. Thus, if an action is blocked in h, it is either no longer contained in $\times_{i\in N}\lambda_i(h',T)$, in which case claim (*ii*) holds with strict inclusion, or it remains blocked in h' by (*iv*), but lies further back in players' memory. Hence, repeated application of Lemma 6.A.2 to the sequence $g_1, g_2, \ldots$ in H with $g_1 = h$ and $g_{k+1} = (\beta(g_k); g_k)$ for all $k \in \mathbb{N}$, yields that a blocked action disappears from memory in at most T steps, in which case the product set of recent actions has become strictly smaller in the weakly decreasing sequence

$$\times_{i\in N}\lambda_i(g_1,T) \supseteq \times_{i\in N}\lambda_i(g_2,T) \supseteq \cdots$$

By (*iii*), the product set remains a prep set. Since there are only finitely many prep sets, it follows that we eventually reach a state g_k without blocked actions. This proves:

Lemma 6.A.3. *Let $h \in H$ be such that $\times_{i\in N}\lambda_i(h,T)$ is a prep set. Either h contains no blocked actions, or the process moves with positive probability in a finite number of steps to a state $h' \in H$ where $\times_{i\in N}\lambda_i(h',T) \subset \times_{i\in N}\lambda_i(h,T)$ is a prep set and h' contains no blocked actions.*

The proof of Step 2 uses so-called drag-to-front operations (Section 6.A.3) and neighbor switches (Section 6.A.4) to establish the following: Given a state $g_1 \in H$ where $\times_{i\in N}\lambda_i(g_1,T)$ is a prep set, the process moves with positive probability in a finite number of steps through a sequence of states $g_1, g_2, \ldots, g_K$ such that

$$\forall k = 1, \ldots, K : \times_{i\in N}\lambda_i(g_k,T) \text{ is a prep set,} \tag{6.6}$$

$$\times_{i\in N}\lambda_i(g_1,T) \supseteq \times_{i\in N}\lambda_i(g_2,T) \supseteq \cdots \supseteq \times_{i\in N}\lambda_i(g_K,T), \tag{6.7}$$

and g_K has the property that for some minimal prep set $Q = \times_{i\in N}Q_i$ and each $i \in N$:

$$\lambda_i(g_K, |Q_i|) = Q_i,$$

that is, for each player $i \in N$, the most recent $|Q_i|$ actions are exactly those in i's component of the minimal prep set Q. If any of the states g_k contains a blocked action, apply Lemma 6.A.3 to move to a state g' where $\times_{i \in N} \lambda_i(g', T) \subset \times_{i \in N} \lambda_i(g_k, T)$ is a prep set and g' contains no blocked actions. Then, we can start the repeated use of drag-to-front operations and neighbor switches anew from g'. Since there are only finitely many prep sets and the prep set $\times_{i \in N} \lambda_i(g', T)$ is strictly contained in $\times_{i \in N} \lambda_i(g_k, T)$, we eventually reach in a finite number of steps a state from which we can apply drag-to-front operations and neighbor switches without ever encountering a state with a blocked action. Hence:

Lemma 6.A.4. *In a sequence of states* $(g_k)_{k=1,\ldots,K}$ *satisfying* (6.6) *and* (6.7), *obtained using drag-to-front operations and neighbor switches, we may assume without loss of generality that none of the states contains a blocked action.*

6.A.3 Drag-to-front operations

Consider a state $h \in H$ containing no blocked actions for which $\times_{i \in N} \lambda_i(h, T)$ is a prep set. Then, by definition, for each $i \in N$, $\beta_i(h) = \tau_i(h)$, the action player i chose T periods ago in state h (see Lemma 6.A.2). Hence, in the successor $(\beta(h); h) = (\tau(h); h)$, this action is dragged to the front of player i's account of recent past play. For easy reference, call the transition from h to $(\beta(h); h) = (\tau(h); h)$ a *drag-to-front operation*.

Suppose some player $j \in N$ has an action $a_j \in \lambda_j(h, T)$ with frequency $f_j(h, a_j) = 1$. Since $T \geq 2|A_j|$ by (6.1), there must be an action $b_j \in \lambda_j(h, T)$ with frequency $f_j(h, b_j) \geq 3$.[5] By Lemma 6.A.4, and using drag-to-front-operations if necessary, we can assume without loss of generality that player j chose b_j exactly T periods ago: $\tau_j(h) = b_j$. For each player $i \in N$, define $\gamma_i(h) \in \lambda_i(h, T)$ as follows:

$$\gamma_i(h) = \begin{cases} \tau_i(h) & \text{if } i \neq j, \\ a_j & \text{if } i = j. \end{cases}$$

Set $h' = (\gamma(h); h)$ with $\gamma(h) = (\gamma_i(h))_{i \in N}$. Recall: (*i*) $\gamma_i(h) \in \lambda_i(h, T)$ for all $i \in N$, (*ii*) $\times_{i \in N} \lambda_i(h, T)$ is a prep set, and (*iii*) no actions in h are blocked; so each $\gamma_i(h)$ is the most recent best reply to a belief $\alpha_{-i} \in \times_{k \in N \setminus \{i\}} \Delta(\lambda_k(h, T))$. By P1 and P2, $P(h, h') > 0$.

By construction, $\times_{i \in N} \lambda_i(h', T) = \times_{i \in N} \lambda_i(h, T)$ remains a prep set. The frequency of the actions of players $i \neq j$ is unaffected: $\forall i \in N \setminus \{j\}, \forall c_i \in \lambda_i(h', T) = \lambda_i(h, T)$:

[5] This motivates the term $2|A_j|$ in the lower bound on T. If the memory length is below this bound, neighbor switches as defined in Section 6.A.4 may cause actions from prep sets to disappear from a player's recollection.

$f_i(h', c_i) = f_i(h, c_i)$. For player j and $c_j \in \lambda_j(h', T) = \lambda_j(h, T)$:

$$f_j(h', c_j) = \begin{cases} f_j(h, c_j) & \text{if } c_j \notin \{a_j, b_j\}, \\ f_j(h, a_j) + 1 = 2 & \text{if } c_j = a_j, \\ f_i(h, b_j) - 1 \geq 2 & \text{if } c_j = b_j. \end{cases}$$

By going from h to h', the number of actions with frequency one has strictly decreased, whereas there is no action with frequency larger than or equal to two whose frequency becomes less than two.

Repeating this process, we eventually reach a state where all actions in the history of recent past play have frequency greater than or equal to two. By Lemma 6.A.3, we may assume that none of its actions is blocked. This proves:

Lemma 6.A.5. *Let $h \in H$ be such that $\times_{i \in N} \lambda_i(h, T)$ is a prep set. Then the process moves with positive probability in a finite number of steps to a state $h' \in H$ with $\times_{i \in N} \lambda_i(h', T) \subseteq \times_{i \in N} \lambda_i(h, T)$ such that*

[C1] *$\times_{i \in N} \lambda_i(h', T)$ is a prep set,*
[C2] *all actions have frequency at least 2: $\forall i \in N, \forall a_i \in \lambda_i(h', T) : f_i(h', a_i) \geq 2$,*
[C3] *h' contains no blocked actions.*

6.A.4 Neighbor switches

Repeatedly applying drag-to-front operations starting in a state $h \in H$ where no actions are blocked and $\times_{i \in N} \lambda_i(h, T)$ is a prep set, we get a sequence of states $g_0, g_1, \dots \in H$ with $g_0 := h$ such that for all players $i \in N$ and all $t \in \mathbb{N}$: $\ell_i(g_t) = \tau_i(g_{t-1})$, i.e., we get a periodic repetition of each player's actions.

Instead, it is possible that some player i chooses his actions in such a way that the process moves to a state in which the order in which player i plays two neighboring actions—say those chosen t and $t+1$ periods ago in state h—is changed, while the others continue to play actions in their given order. For instance, the process may move from Figure 6.2(a) to Figure 6.2(e), where player i's order of actions b and c, chosen 2 and 3 periods ago in Fig 6.2(a), respectively, is reversed while the order of actions of players $j \neq i$ is unchanged. In Figure 6.2, the length of recent past play T is 4; actions chosen during the most recent four periods are contained in the boxed part of the table; actions outside the boxes have disappeared from recent past play. For instance, in Figure 6.2(c), player i chose c five periods ago, d six periods ago. Since $T = 4$, these actions are no longer part of recent past play.

(a)

player i:	a	b	c	d
player j:	α	β	γ	δ

(b)

player i:	d	a	b	c	d
player j:	δ	α	β	γ	δ

(c)

player i:	b	d	a	b	c	d
player j:	γ	δ	α	β	γ	δ

(d)

player i:	c	b	d	a	b	c	d
player j:	β	γ	δ	α	β	γ	δ

(e)

player i:	a	c	b	d	a	b	c	d
player j:	α	β	γ	δ	α	β	γ	δ

Figure 6.2. Switch i's actions b and c, keeping those of players $j \neq i$ in the same order.

The idea is simple:[6] use drag-to-front operations until the actions to be switched are those chosen $T-1$ and T periods ago (the transition from Figure 6.2(a) to Figure 6.2(b)); in the next two periods, let players $j \neq i$ continue with drag-to-front operations, while player i chooses the actions that are to be switched in reverse order (in going from Figure 6.2(b) to Figure 6.2(c), i chooses b instead of c, in going from the Figure 6.2(c) to Figure 6.2(d), i chooses c instead of b). Finally, use drag-to-front operations until the switched actions are again at coordinates t and $t+1$ in the recent past play (the transition from Figure 6.2(d) to Figure 6.2(e)). Formally:

Lemma 6.A.6. *Let $h \in H$ satisfy* [C1]-[C3]. *Let $i \in N, t \in \{1, \dots, T-1\}$. Assuming without loss of generality (Lemma 6.A.4) that we encounter no blocked actions, the process moves with positive probability in T steps to a state $h' \in H$ satisfying* [C1] - [C3] *and in which $a_j(h', k) = a_j(h, k)$ if $j = i$ and $k \notin \{t, t+1\}$, or if $j \neq i$, whereas $a_i(h', t) = a_i(h, t+1)$ and $a_i(h', t+1) = a_i(h, t)$.*

Proof. For notational convenience, let a_i and b_i be the actions player i chose $t+1$ and t periods ago in h, respectively. Performing $T-t-1$ drag-to-front operations, we reach a state g_1 satisfying [C1] - [C3] in which a_i is the action i chose T periods ago and b_i the action he chose $T-1$ periods ago.

[6] Figure 6.2 is for illustration only; it is assumed that all steps described there are feasible.

Construct a successor g_2 of g_1 as follows: for each $j \in N \setminus \{i\}$, set $s_j^1 = \tau_j(g_1)$ and set $s_i^1 = b_i$. Define $g_2 = (s^1; g_1)$, where $s^1 = (s_j^1)_{j \in N}$.

Construct a successor g_3 of g_2 as follows: for each $j \in N \setminus \{i\}$, set $s_j^2 = \tau_j(g_2)$ and set $s_i^2 = a_i$. Define $g_3 = (s^2; g_2)$, where $s^2 = (s_j^2)_{j \in N}$.

For players $j \neq i$, these two steps involve simple drag-to-front operations. For player i it involves reversing the order: in going from g_1 to g_2, i chooses b_i, in going from g_2 to g_3, i chooses a_i, rather than playing first a_i, then b_i.

As $\times_{i \in N} \lambda_i(g_1, T)$ is a prep set and no actions are blocked in g_1, it follows from P1 and P2 that $P(g_1, g_2) > 0$. Moreover, as all actions in h have frequency at least 2, we have that $\lambda_i(g_1, T) = \lambda_i(g_2, T)$ for all $i \in N$. Hence, also $\times_{i \in N} \lambda_i(g_2, T)$ is a prep set. By Lemma 6.A.4 we may assume that g_2 contains no blocked actions. Hence, also $P(g_2, g_3) > 0$. Moreover, it is easy to see that frequencies in g_3 are identical to frequencies in g_1, i.e., at least equal to 2. We can thus conclude that also g_3 satisfies [C1] - [C3].

In g_3, the two actions that are played most recently are a_i and b_i, respectively. Thus, performing $t - 1$ drag-to-front operations leads to the desired state h'. □

6.A.5 Proof of Steps 2 to 4

Step 2: Let $h_1 \in H$ be such that $\times_{i \in N} \lambda_i(h_1, T)$ is a prep set. The process moves with positive probability in a finite number of steps to a state $h_2 \in H$ where $\times_{i \in N} \lambda_i(h_2, T)$ is a minimal prep set.

Proof of Step 2: By Lemma 6.A.5, the process moves with positive probability in a finite number of steps from h_1 to a state $g \in H$ satisfying [C1] - [C3]. Let $Q = \times_{i \in N} Q_i \subseteq \times_{i \in N} \lambda_i(g, T)$ be a minimal prep set. Assuming without loss of generality (Lemma 6.A.4) that from g onward we do not encounter blocked actions, Lemma 6.A.6 allows us to perform neighbor switches. Every permutation of a finite set can be obtained by a chain of neighbor switches; thus, repeated application of Lemma 6.A.6 yields that the process moves in a finite number of steps to a state $g_0 \in H$ with the property that for each player $i \in N$, $\lambda_i(g_0, |Q_i|) = Q_i$, that is, for each player $i \in N$, the most recent $|Q_i|$ actions in g_0 are exactly those in i's component of the minimal prep set Q.

For each $k \in \mathbb{N}$, let $g_k := ((a_i(g_{k-1}, |Q_i|))_{i \in N}; g_{k-1}) \in H$, i.e., g_k is the successor of g_{k-1} obtained by letting each player $i \in N$ play the action he chose $|Q_i|$ periods ago in g_{k-1}. Recalling that Q is a minimal prep set, a simple inductive proof establishes that for all $k \in \mathbb{N}$ it holds that $P(g_{k-1}, g_k) > 0$ and for all players $i \in N$ we have

$$\lambda_i(g_k, \min\{|Q_i| + k, T\}) = Q_i.$$

Set $k = T$ to find that $\times_{i \in N} \lambda_i(g_T, T) = Q$. Taking $h_2 := g_T$ finishes the proof of Step 2.

Step 3: Let $h_2 \in H$ be such that $Q = \times_{i \in N} \lambda_i(h_2, T)$ is a minimal prep set. After reaching h_2, all action profiles that are played with positive probability lie in Q:

$$\forall k \in \mathbb{N}, \forall h \in H : P^k(h_2, h) > 0 \Rightarrow \ell(h) \in Q. \tag{6.8}$$

Proof of Step 3: By P1 and P2, players always base beliefs on the actions played in the last T periods and choose the most recent best reply to such beliefs. In h_2, their account of recent play $\times_{i \in N} \lambda_i(h_2, T)$ equals the minimal prep set Q, which by definition contains at least one best reply to whatever belief a player may have about other players' choices from Q. Hence, by induction, the actions from minimal prep set Q will always be fresher in players' recollection of past play than actions outside Q, i.e., beliefs and best replies to these beliefs will, by P1 and P2, always have support in Q. Formally, for all $k \in \mathbb{N}$ and $h \in H$:

$$\text{if } P^k(h_2, h) > 0, \text{ then } \times_{i \in N} \lambda_i(h, T + k) = Q,$$

and hence

$$\times_{i \in N} \lambda_i(h, T) \subseteq Q.$$

In particular, this means $\ell(h) \in Q$, i.e., (6.8) holds.

Step 4: For every state $h_0 \in H$, the process eventually reaches a state $h_2 \in H$ satisfying the conditions in Step 2, i.e., where according to Step 3 play settles down in a minimal prep set.

Proof of Step 4: Call two states $h = (a^1, \ldots, a^L)$ and $g = (b^1, \ldots, b^K)$ in H equivalent, denoted $h \sim g$, if they have the same account of recent past play and the same order in which each player i's actions are encountered:

$$h \sim g \Leftrightarrow \begin{cases} (a^1, \ldots, a^T) = (b^1, \ldots, b^T), \\ o_{i,h} = o_{i,g} \text{ for all } i \in N. \end{cases}$$

Notice that $\sim$ is an equivalence relation on H; for each $h \in H$, let $[h] = \{h' \in H : h \sim h'\}$ be the equivalence class containing h. Recall from Section 6.3 that in each state $h \in H$, if the sequence of action profiles from the past T periods is $(a^1, \dots, a^T) \in A^T$, then, firstly, player i's beliefs α_{-i} are drawn from a probability distribution $\mathbb{P}_{(i,(a^1,\dots,a^T))}$ and, secondly, his response is (whenever possible) the most recent best reply to this belief or (otherwise) drawn from a probability distribution $\mathbb{P}_{\alpha_{-i}}$ over his best replies. Thus, player i's choice behavior is the same in two equivalent states. Since there are only finitely many elements in A^T and N, it follows that the set of positive transition probabilities $\{P(h,h') \mid h, h' \in H, P(h,h') > 0\}$ is a finite set. Let $\varepsilon > 0$ be its minimum.

By Steps 1 to 3, it is possible, from any history $h_0 \in H$, to reach a state $h_2 \in H$ in an absorbing set where play settles down in a minimal prep set in a finite number of steps, say $k(h_0) \in \mathbb{N}$. By definition of equivalence, $k(h) = k(h_0)$ for all $h \in [h_0]$: the set $\{k(h_0) \mid h_0 \in H\}$ is finite. Let $\kappa \in \mathbb{N}$ be its minimum.

By definition of ε and κ, the probability of entering an absorbing set where play settles down in a minimal prep set in at most κ steps is at least ε^κ from any state. Hence, the probability of not reaching an absorbing set in κ steps is at most $1 - \varepsilon^\kappa$, which is less than 1. So the probability of not reaching an absorbing set in $k\kappa$ steps is less than or equal to $(1 - \varepsilon^\kappa)^k$, which goes to zero as k goes to infinity. □

6.B Proof of Proposition 6.5.3

The set inclusion $\mathscr{P} \subset \widetilde{\mathscr{P}}$ was established in Remark 6.5.1 and Example 6.5.2. To establish the convergence part, the proof of Theorem 6.4.1 in Appendix 6.A applies with minor changes to P_R as well:

– condition (α) guarantees that Steps 1 and 2 hold without change,

– condition (β) guarantees that Step 3 holds without change,

– by (6.2), there are only finitely many different functions in $R = (R_{i,h})_{i \in N, h \in H}$, so the equivalence relation in Step 4 is well-defined and there are again finitely many equivalence classes; hence, also Step 4 holds. □

7 An axiomatization of minimal prep sets

Summary

In the previous chapter, it was shown that behaviorally plausible adjustment processes settle down in minimal prep sets. This chapter, which is based on Voorneveld, Kets, and Norde (2005, 2006), investigates this set-valued solution concept further by providing an axiomatic characterization. We show that the concept satisfies the axiom of consistency. We also clarify the relation between minimal prep sets and minimal curb sets.

7.1 Introduction

In the previous chapter, we found that learning processes in which players display a recency bias converge to a minimal prep set (Voorneveld, 2004). Here, we study the properties of this set-valued solution concept by providing an axiomatic characterization of this concept and of the closely related concept of a minimal curb set (Basu and Weibull, 1991). We provide an axiomatization of these solution concepts in terms of consistency and other axioms, thus clarifying the close relation between the two concepts.

Consistency is a central axiom in game theory. The notion of consistency for solutions of noncooperative games was introduced by Peleg and Tijs (1996) and Peleg et al. (1996). Consistency essentially requires that if a nonempty set of players commits to playing according to a certain solution, the remaining players in the reduced game should not have an incentive to deviate from it. This appears to be a minimal requirement on a solution concept (see also Aumann (1987, pp. 478–479): given that others play the game according to a certain solution, the solution concept should recommend you to do the same.

While Norde et al. (1996) proves that the unique point-valued solution concept for the set of strategic games that satisfies consistency, utility maximizing behavior in one-player games and nonemptiness, is the Nash equilibrium concept, Dufwenberg et al. (2001) show by means of examples that a transition to set-valued solution concepts overcomes consistency problems. They show that there is a multiplicity of consistent set-valued solution concepts that satisfy nonemptiness and recommend utility maximization in one-player games. Among these concepts are minimal prep sets and minimal curb sets.

Building on these papers, which strive for characterizations of solution concepts in terms of consistency and other properties or axioms, we provide a similar axiomatization of minimal prep sets and minimal curb sets. Section 7.2 describes properties of set-valued solution concepts. It is shown that the set-valued solution concept that assigns to each game its collection of minimal prep sets satisfies these properties (Proposition 7.2.3); indeed, it is the only one (Theorem. 7.3.1). Moreover, the properties are logically independent (Proposition 7.3.2). In Section 7.4, we give an axiomatization of minimal curb sets, and discuss the relation between minimal prep sets and minimal curb sets. In addition, we discuss some variants and extensions of the main result.

7.2 Properties of set-valued solution concepts

Recall the definition of a (finite strategic) game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ (Section 2.1.1), where N is a finite, nonempty set of *players*, and each player $i \in N$ is endowed with a finite, nonempty set A_i of *actions* and a *(von Neumann Morgenstern) utility function* on the set of pure strategy profiles $A = \times_{j\in N} A_j$. The set of all finite strategic games is denoted by Γ. The set of mixed strategies of player $i \in N$ with support in $X_i \subseteq A_i$ is denoted by $\Delta(X_i)$. Payoffs are extended to mixed strategies in the usual way. As usual, (a_i, α_{-i}) is the profile of strategies where player $i \in N$ plays $a_i \in A_i$ and his opponents play according to the mixed strategy profile $\alpha_{-i} = (\alpha_j)_{j\in N\setminus\{i\}} \in \times_{j\in N\setminus\{i\}} \Delta(A_j)$. For $i \in N$ and $\alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(A_j)$,

$$BR_i(\alpha_{-i}) = \{a_i \in A_i \mid \forall b_i \in A_i : u_i(a_i, \alpha_{-i}) \geq u_i(b_i, \alpha_{-i})\}.$$

is the set of pure best replies of player i against α_{-i}.

Definition 7.2.1. *A* prep set *of a game* $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$ *is a nonempty product set* $Q = \times_{i\in N} Q_i \subseteq A$ *such that for each* $i \in N$ *and each belief* $\alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(Q_j)$ *of player* i*, the set* Q_i *contains* at least one *best response of player* i *against his belief:*

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(Q_j) : \quad BR_i(\alpha_{-i}) \cap Q_i \neq \emptyset.$$

A prep set Q *is* minimal *if no prep set is a proper subset of* Q*.*

Definition 7.2.2. *A* curb set *of a game* $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ *is a nonempty product set* $Q = \times_{i \in N} Q_i \subseteq A$ *such that for each* $i \in N$ *and each belief* $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(Q_j)$ *of player* i*, the set* Q_i *contains* all *best responses of player* i *against his belief:*

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(Q_j) : \quad BR_i(\alpha_{-i}) \subseteq Q_i.$$

A curb set Q *is* minimal *if no curb set is a proper subset of* Q*.*

The set-valued solution concept that assigns to each game its collection of minimal prep sets is denoted by min-prep, and the set-valued solution concept that assigns to each game its collection of minimal curb sets is denoted by min-curb.

We provide properties of set-valued solution concepts and show that min-prep satisfies these properties. Section 7.4 discusses the properties of minimal curb sets, and discusses some variants. Throughout this section, φ^s is an arbitrary set-valued solution concept on the set of all finite strategic games Γ.

The first three properties we discuss here are well known from Peleg and Tijs (1996), Peleg et al. (1996), and Norde et al. (1996) for point-valued solutions like the Nash equilibrium concept. We restate them here for set-valued solution concepts. The property of *nonemptiness* requires that the solution concept assigns to each game a nonempty collection of solutions. *One-person rationality* requires that in one-player games, the solution simply maximizes the player's utility.

Nonemptiness: $\varphi^s(G) \neq \emptyset$ for each $G \in \Gamma$.
One-person rationality: for each one-player game $G = \langle \{i\}, A_i, u_i \rangle \in \Gamma$ it holds that $\varphi^s(G) = \{\{b_i\} \mid b_i \in \arg\max_{a_i \in A_i} u_i(a_i)\}$.

The idea behind *consistency* is that if some players commit to playing according to a certain solution, the remaining players should have an incentive to do so as well. This requires appropriate ways to model (*i*) the reduced game that arises if some players commit to a certain behavior, (*ii*) the absence of incentives to deviate, i.e., the statement that the solution of the original game gives rise to a solution of the reduced game.

Different models of these issues yield different forms of *consistency*. In this chapter we use the notion of reduced games as defined by Peleg and Tijs (1996), Peleg et al. (1996), and Norde et al. (1996): given a game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ with at least two players and a mixed strategy profile $\alpha \in \times_{i \in N} \Delta(A_i)$, fix a coalition $S \subset N, S \neq \emptyset$, and suppose that the players in $N \setminus S$ commit to playing their part of α. The *reduced game* with respect to S and α is the game $G^{S,\alpha} = \langle S, (A_i)_{i \in S}, (v_i)_{i \in S} \rangle \in \Gamma$ where only players $i \in S$ choose from their set of pure strategies A_i, while their payoff functions reduce to $v_i : \times_{j \in S} A_j \to \mathbb{R}$ defined as $v_i(\cdot) = u_i(\cdot, \alpha_{N \setminus S})$, i.e., the payoff in the original game, given that members of $N \setminus S$ play $\alpha_{N \setminus S} = (\alpha_j)_{j \in N \setminus S}$ in accordance with α.

The next step models the statement that a solution of the original game gives rise to a solution of the reduced game. Consider a solution $Q \in \varphi^s(G)$ of $G \in \Gamma$. Playing according to Q implies restricting attention to mixed strategy profiles $\alpha \in \times_{i \in N} \Delta(Q_i)$. Fix some coalition $S \subset N, S \neq \emptyset$, of players and suppose that the members of $N \setminus S$ commit to such a strategy profile α, thus yielding the reduced game $G^{S,\alpha}$. *Consistency* now requires that the initial solution $Q \in \varphi^s(G)$ yields a solution of the reduced game in the following sense: the reduced game $G^{S,\alpha}$ has a solution in $\times_{j \in S} Q_j$, the relevant part of $Q \in \varphi^s(G)$.

Consistency: for each $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$, each $Q = \times_{i \in N} Q_i \in \varphi^s(G)$, each $\alpha \in \times_{i \in N} \Delta(Q_i)$, each $S \subset N, S \neq \emptyset$, there is a solution $Y \in \varphi^s(G^{S,\alpha})$ with $Y \subseteq \times_{j \in S} Q_j$.

The other properties are specific for set-valued solution concepts, but remain standard.

Nonnestedness: for each $G \in \Gamma$, there are no $Q, Y \in \varphi^s(G)$ with $Q \subset Y$.

Many common set-valued solution concepts satisfy *nonnestedness*, including those defined by product sets of actions which (*i*) survive some iterated elimination process, for instance of strictly/weakly dominated actions, or, in the case of rationalizability, of never-best replies, or (*ii*) are minimal or maximal sets with some desirable property. This includes persistent retracts (so-called minimal absorbing retracts, see Kalai and Samet, 1984, pp. 134–135), the product set of all minimax/maximin actions in two-person zero-sum games, the product set of all rationalizable actions (the so-called maximal tight curb set, see Basu and Weibull, 1991, p. 145), or the largest consistent set of Chwe (1994, pp. 313–318); his use of the word "consistent" is unrelated to our notion of *consistency*.

The next property, *satisfaction*, uses the notion of a subgame of a game. The *subgame* obtained from G by restricting the action set of each player $i \in N$ to a subset $Q_i \subseteq A_i$ is denoted by $\langle N, (Q_i)_{i \in N}, (u_i)_{i \in N} \rangle$. Note that this is with minor abuse of notation as we restrict the domain of the payoff functions u_i to $\times_{i \in N} Q_i$. The property of *satisfaction* is a simple revealed-preference property. A product set of strategies is called satisfactory, given the solution concept φ^s, if players can credibly commit to playing actions from that set if they believe that others do so: it always contains a solution of the associated reduced game. Given such credible commitment, *satisfaction*[1] states that a way of finding solutions of the original game is to solve the subgame restricted to a satisfactory set.

Formally, consider a game $G \in \Gamma$ with at least two players and a product set $Q = \times_{i \in N} Q_i \subseteq A$. Such a set is called *satisfactory under* φ^s if for each $\alpha \in \times_{i \in N} \Delta(Q_i)$ and each $S \subset N, S \neq \emptyset$, there exists a $Y \in \varphi^s(G^{S,\alpha})$ with $Y \subseteq \times_{j \in S} Q_j$.

> **Satisfaction:** for each $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ with $|N| \geq 2$ and each $Q \subseteq A$ which is satisfactory under φ^s, one has $\varphi^s(\langle N, (Q_i)_{i \in N}, (u_i)_{i \in N} \rangle) \subseteq \varphi^s(G)$.

This property is reminiscent of the converse consistency axiom of Peleg and Tijs (1996) and Peleg et al. (1996), which roughly states that if a solution candidate always yields a solution in the associated reduced games, it is indeed a solution of the original game. Note that *satisfaction* is much weaker, as satisfactory sets need not be contained in the solution of the game.

Proposition 7.2.3. *The set-valued solution concept* min-prep *satisfies nonemptiness, one-person rationality, consistency, nonnestedness, and satisfaction.*

Proof. *Nonemptiness*: See Proposition 2.1.7.

One-person rationality: Let $G = \langle \{i\}, A_i, u_i \rangle \in \Gamma$ be a one-player game. In a one-player game, the set of best responses is simply the set of maximizers of the utility function. Hence, $Q_i \subseteq A_i$ is a prep set of G if and only if $\arg\max_{a_i \in A_i} u_i(a_i) \cap Q_i \neq \emptyset$; it is a minimal prep set of G if and only if $Q_i = \{b_i\}$ for some $b_i \in \arg\max_{a_i \in A_i} u_i(a_i)$. Hence, $\text{min-prep}(G) = \{\{b_i\} \mid b_i \in \arg\max_{a_i \in A_i} u_i(a_i)\}$.

Consistency: Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$, $Q = \times_{i \in N} Q_i \in \text{min-prep}(G)$, $\alpha \in \times_{i \in N} \Delta(Q_i)$, and $S \subset N, S \neq \emptyset$. We need to show that there is a $Y \in \text{min-prep}(G^{S,\alpha})$ with $Y \subseteq \times_{j \in S} Q_j$. Since $Q \in \text{min-prep}(G)$, it follows that $\times_{j \in S} Q_j \in \text{prep}(G^{S,\alpha})$. Since $\times_{j \in S} Q_j \in \text{prep}(G^{S,\alpha})$ and there are only finitely many prep sets in $G^{S,\alpha}$, it contains a minimal one. Hence, there is a $Y \in \text{min-prep}(G^{S,\alpha})$ with $Y \subseteq \times_{j \in S} Q_j$.

Nonnestedness: Holds by minimality.

Satisfaction: Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ with $|N| \geq 2$. Let $Q \subseteq A$ be a satisfactory set under min-prep. To show:

$$\text{min-prep}(\langle N, (Q_i)_{i \in N}, (u_i)_{i \in N} \rangle) \subseteq \text{min-prep}(G). \tag{7.1}$$

We first show that $Q \in \text{prep}(G)$. Let $i \in N$ and $\alpha \in \times_{j \in N} \Delta(Q_j)$. Since Q is a satisfactory set under min-prep, there is a $Y \in \text{min-prep}(G^{\{i\},\alpha})$ with $Y \subseteq Q_i$. But

[1] The adjective "satisfactory" describes a property of product sets, the noun "*satisfaction*" describes a property of a solution concept.

$G^{\{i\},\alpha}$ is the one-player game $\langle\{i\}, A_i, v_i\rangle \in \Gamma$ with $v_i(a_i) = u_i(a_i, \alpha_{-i})$ for each $a_i \in A_i$. Hence, min-prep$(G^{\{i\},\alpha}) = \{\{b_i\} \mid b_i \in \arg\max_{a_i \in A_i} u_i(a_i, \alpha_{-i})\}$, so $Y = \{b_i\}$ for some $b_i \in \arg\max_{a_i \in A_i} u_i(a_i, \alpha_{-i})$. As $Y \subseteq Q_i$, Q_i contains at least one best reply to the belief $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(Q_j)$. Since this holds for arbitrary $i \in N$ and $\alpha \in \times_{j \in N} \Delta(Q_j)$, it holds by definition that $Q \in$ prep(G).

We now prove (7.1) by contradiction. Let $Y \in$ min-prep$(\langle N, (Q_i)_{i \in N}, (u_i)_{i \in N}\rangle)$. Since $Q \in$ prep(G), we also have $Y \in$ prep(G). If $Y \notin$ min-prep(G), there is a $Z \in$ min-prep(G) with $Z \subset Y$. But since $Z \in$ min-prep(G), it is also a prep set of the subgame $G' = \langle N, (Q_i)_{i \in N}, (u_i)_{i \in N}\rangle$, contradicting that $Y \in$ min-prep(G'). Conclude that (7.1) holds. □

7.3 Axiomatization

In this section, we show that min-prep is the unique solution concept satisfying the properties in Section 7.2 and that these properties are logically independent.

Theorem 7.3.1. *The unique set-valued solution concept on Γ satisfying nonemptiness, one-person rationality, consistency, nonnestedness, and satisfaction is* min-prep*.*

Proof. Proposition 7.2.3 shows that min-prep satisfies the properties. Let φ^s be a set-valued solution concept on Γ that also satisfies them. We need to show that $\varphi^s(G) =$ min-prep(G) for each $G \in \Gamma$. We show this by induction on the number of players. In a one-player game $G = \langle\{i\}, A_i, u_i\rangle \in \Gamma$, it follows from *one-person rationality* of φ^s and min-prep that

$$\varphi^s(G) = \text{min-prep}(G) = \left\{ \{b_i\} \middle| b_i \in \arg\max_{a_i \in A_i} u_i(a_i) \right\}.$$

Next, let $n \in \mathbb{N}$ and assume that φ^s and min-prep coincide on all games in Γ with at most n players. Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N}\rangle \in \Gamma$ have $n+1$ players.

Step 1: $\varphi^s(G) \subseteq$ prep(G).

Proof of Step 1: Let $Q \in \varphi^s(G), i \in N$, and $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(Q_j)$. We need to show that $BR_i(\alpha_{-i}) \cap Q_i \neq \emptyset$. Let $\beta \in \times_{j \in N} \Delta(Q_j)$ be a mixed strategy profile with $\beta_{-i} = \alpha_{-i}$. By *consistency* of φ^s, there is a solution $Y \in \varphi^s(G^{\{i\},\beta})$ with $Y \subseteq Q_i$. The game $G^{\{i\},\beta}$ is the one-player game $\langle\{i\}, A_i, v_i\rangle \in \Gamma$ with $v_i(a_i) = u_i(a_i, \beta_{-i}) = u_i(a_i, \alpha_{-i})$ for each

$a_i \in A_i$. By *one-person rationality* of φ^s, it follows that

$$\varphi^s(G^{\{i\},\beta}) = \left\{\{b_i\}\middle| b_i \in \arg\max_{a_i \in A_i} v_i(a_i)\right\} = \left\{\{b_i\}\middle| b_i \in \arg\max_{a_i \in A_i} u_i(a_i, \alpha_{-i})\right\},$$

i.e., $Y = \{b_i\}$ for some $b_i \in \arg\max_{a_i \in A_i} u_i(a_i, \alpha_{-i})$. As $Y \subseteq Q_i$, $BR_i(\alpha_{-i}) \cap Q_i \neq \emptyset$ as we had to show.

Step 2: If $Q \in$ min-prep(G), then Q is a satisfactory set under φ^s.

Proof of Step 2: Let $Q \in$ min-prep(G), $\alpha \in \times_{i \in N} \Delta(Q_i)$, and $S \subset N, S \neq \emptyset$. By induction, $\varphi^s(G^{S,\alpha}) =$ min-prep($G^{S,\alpha}$). By *consistency* of min-prep, there is a $Y \in$ min-prep($G^{S,\alpha}$) with $Y \subseteq \times_{i \in S} Q_i$. Combining these two results, we find that there is a $Y \in \varphi^s(G^{S,\alpha})$ with $Y \subseteq \times_{i \in S} Q_i$. Hence, Q is a satisfactory set under φ^s.

Step 3: If $Q \in$ min-prep(G), then there is a $Y \in \varphi^s(G)$ with $Y \subseteq Q$.

Proof of Step 3: Let $Q \in$ min-prep(G). By Step 2, Q is a satisfactory set under φ^s. Since φ^s satisfies *nonemptiness* and *satisfaction*, it follows that

$$\emptyset \neq \varphi^s(\langle N, (Q_i)_{i \in N}, (u_i)_{i \in N}\rangle) \subseteq \varphi^s(G). \tag{7.2}$$

So let $Y \in \varphi^s(\langle N, (Q_i)_{i \in N}, (u_i)_{i \in N}\rangle)$. Then $Y \subseteq Q$, and by (7.2), $Y \in \varphi^s(G)$.

Step 4: $\varphi^s(G) \subseteq$ min-prep(G).

Proof of Step 4: Let $Q \in \varphi^s(G)$. By Step 1, $Q \in$ prep(G). Suppose $Q \notin$ min-prep(G): there is a $Y \in$ min-prep(G) with $Y \subset Q$. By Step 3, there is a $Z \in \varphi^s(G)$ with $Z \subseteq Y$. But since $Z \subseteq Y \subset Q$ and $Q, Z \in \varphi^s(G)$, we have a contradiction with the assumption that φ^s is *nonnested*. Conclude that $Q \in$ min-prep(G).

Step 5: min-prep(G) $\subseteq \varphi^s(G)$.

Proof of Step 5: Let $Q \in$ min-prep(G). By Step 3, there is a $Y \subseteq Q$ with $Y \in \varphi^s(G)$. By Step 1, $Y \in$ prep(G). Since $Q \in$ min-prep(G) and $Y \subseteq Q$ is a prep set, it follows that $Y = Q$, i.e., $Q = Y \in \varphi^s(G)$.

Combining Steps 4 and 5, conclude that $\varphi^s(G) =$ min-prep(G) also for the $(n+1)$-player game G. Hence, by induction, $\varphi^s(G) =$ min-prep(G) for each $G \in \Gamma$. □

Proposition 7.3.2. *The axioms in Theorem 7.3.1 are logically independent.*

We show this by presenting five set-valued solution concepts, each violating exactly one of the five axioms in Theorem 7.3.1. Since the verification that these concepts satisfy the given properties proceeds along the same lines as the proof of Proposition 7.2.3, we only show explicitly which axiom is violated. Solution concepts φ_1^s to φ_5^s are defined, for each game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$, as follows:

$$\varphi_1^s(G) = \begin{cases} \text{min-prep}(G) & \text{if } G \text{ is a one-player game,} \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\varphi_2^s(G) = \text{min-curb}(G).$$

$$\varphi_3^s(G) = \begin{cases} \text{min-prep}(G) & \text{if } G \text{ is a one-player game,} \\ \{\times_{i\in N}\{a_i\} \mid \forall i \in N : a_i \in A_i\} & \text{otherwise.} \end{cases}$$

$$\varphi_4^s(G) = \begin{cases} \text{min-prep}(G) & \text{if } G \text{ is a one-player game,} \\ \text{prep}(G) & \text{otherwise.} \end{cases}$$

$$\varphi_5^s(G) = \begin{cases} \text{min-prep}(G) & \text{if } G \text{ is a one-player game,} \\ \{A\} & \text{otherwise.} \end{cases}$$

The solution concept φ_1^s satisfies all properties in Theorem 7.3.1, except *nonemptiness*: $\varphi_1^s(G) = \emptyset$ for each game $G \in \Gamma$ with two or more players.

The solution concept φ_2^s satisfies all properties in Theorem 7.3.1, except *one-person rationality*: in the one-player game $G = \langle\{1\}, \{a, b\}, u_1\rangle$ with $u_1(a) = u_1(b)$, we have

$$\varphi_2^s(G) = \text{min-curb}(G) = \{\{a, b\}\} \neq \{\{a\}, \{b\}\} = \left\{\{d\} \;\middle|\; d \in \arg\max_{c\in\{a,b\}} u_1(c)\right\}.$$

The solution concept φ_3^s satisfies all properties in Theorem 7.3.1, except *consistency*. In the game G in Figure 7.1, $Q = \{B\} \times \{L\} \in \varphi_3^s(G)$. Consider the belief (B, L) in which player 1 chooses B with probability one and player 2 chooses L with probability one. In the reduced game $G^{\{1\},(B,L)} = \langle\{1\}, \{T, B\}, v_1\rangle$ with $v_1(T) = 1$ and $v_1(B) = 0$, we have

$$\varphi_3^s\Big(G^{\{1\},(B,L)}\Big) = \text{min-prep}(G^{\{1\},(B,L)}) = \{\{T\}\},$$

so $Q_1 = \{B\}$ does not contain a solution of the reduced game $G^{\{1\},(B,L)}$.

The solution concept φ_4^s satisfies all properties in Theorem 7.3.1, except *nonnestedness*: in the game G in Figure 7.1, we have $\varphi_4^s(G) = \text{prep}(G) = \{\{T\} \times \{L\}, \{B\} \times \{R\}, \{T\} \times \{L, R\}, \{T, B\} \times \{L\}, \{T, B\} \times \{L, R\}\}$ with for instance $\{T\} \times \{L\} \subset \{T, B\} \times \{L, R\}$.

	L	R
T	1,1	0,0
B	0,0	0,0

Figure 7.1. A simple two-player game G.

The solution concept φ_5^s satisfies all properties in Theorem 7.3.1, except *satisfaction*: in the two-player game G in Figure 7.1, $\{T\} \times \{L\}$ is a satisfactory set under φ_5^s, but in the subgame G' restricted to $\{T\} \times \{L\}$, we have $\varphi_5^s(G') = \{\{T\} \times \{L\}\} \not\subseteq \{\{T, B\} \times \{L, R\}\} = \varphi_5^s(G)$.

7.4 Variants and extensions

(a) In Theorem 7.3.1, *nonnestedness* can be replaced by the following property:

Decisiveness: for each $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$ and $Q \in \varphi^s(G)$:

$$\varphi^s(\langle N, (Q_i)_{i\in N}, (u_i)_{i\in N}\rangle) = \{Q\}.$$

The intuition behind *decisiveness* is that the solution concept takes some argument to its logical conclusion: given a solution Q of a game, the solution of the subgame restricted to Q is not refined further. Note that min-prep satisfies *decisiveness*. *Nonnestedness* is used only in Step 4 of Theorem 7.3.1, the proof of which now is as follows: Let $Q \in \varphi^s(G)$. By Step 1, $Q \in \text{prep}(G)$. Let $Y \in \text{min-prep}(G)$ with $Y \subseteq Q$. Then also $Y \in \text{min-prep}(\langle N, (Q_i)_{i\in N}, (u_i)_{i\in N}\rangle)$. By Step 3 applied to the subgame $\langle N, (Q_i)_{i\in N}, (u_i)_{i\in N}\rangle$, there is a $Z \in \varphi^s(\langle N, (Q_i)_{i\in N}, (u_i)_{i\in N}\rangle)$ with $Z \subseteq Y$. *Decisiveness* of φ^s implies that $\varphi^s(\langle N, (Q_i)_{i\in N}, (u_i)_{i\in N}\rangle) = \{Q\}$. Conclude that $Q = Z \subseteq Y \subseteq Q$, i.e., $Q = Y \in \text{min-prep}(G)$, proving Step 4.

The set-valued solution concepts φ_1^s to φ_5^s can be used to show that the new axiom system, with *decisiveness* instead of *nonnestedness*, uses logically independent properties; see Figure 7.3 for a summary.

(b) Since most of the literature on minimal prep sets and minimal curb sets concerns mixed extensions of finite strategic games, we have taken this set to be our domain Γ. The assumption that the games are finite is not necessary: we essentially need Γ to be closed with respect to certain subgames and reduced games, and that each game in Γ has a nonempty collection of minimal prep sets. In particular, defining prep sets and the properties in Section 7.2 in terms of product sets $Q = \times_{i \in N} Q_i$ where each component Q_i is a nonempty *compact* set of pure strategies, our analysis carries through also on the domain of games where each strategy space is assumed to be compact Hausdorff and utility functions are sufficiently measurable and upper semicontinuous on the own strategy space, the domain of games on which Voorneveld (2004) establishes existence of minimal prep sets.

(c) Rationality requires decision makers in one-player games to choose utility maximizing actions. That is the motivation behind the standard *one-person rationality* axiom in the consistency literature. For set-valued solution concepts, it matters whether one considers the utility-maximizing actions separately, as was done in the definition of *one-person rationality*, or that one collects them in a single set. For instance, in the one-player game $G = \langle \{1\}, \{a, b\}, u_1 \rangle$ with $u_1(a) = u_1(b)$, we have min-prep$(G) = \{\{a\}, \{b\}\}$, whereas min-curb$(G) = \{\{a, b\}\}$: while prep sets require the presence of at least one best reply, curb sets require all "best replies" to be present. An intuitive modification of the *one-person rationality* axiom in Section 7.2 would therefore be:

$$\text{For each } G = \langle \{i\}, A_i, u_i \rangle \in \Gamma : \quad \varphi^s(G) = \left\{ \arg\max_{a_i \in A_i} u_i(a_i) \right\}. \tag{7.3}$$

Rewriting our earlier results yields an axiomatization of min-curb:

Theorem 7.4.1. *The unique set-valued solution concept on* Γ *satisfying nonemptiness, one-person rationality as in* (7.3), *consistency, nonnestedness, and satisfaction is* min-curb.

Proposition 7.4.2. *The axioms in Theorem 7.4.1 are logically independent.*

The proofs of these results are virtually identical to those of Propositions 7.2.3, 7.3.2, and Theorem 7.3.1 by interchanging, firstly, prep and curb and, secondly, min-prep and min-curb, and are therefore omitted. In analogy with the remark under **(a)**, *nonnestedness* can be replaced with *decisiveness*; the axioms remain logically independent. In fact, the paper Voorneveld et al. (2005, 2006), on which this chapter is based, presents all proofs for minimal curb sets, rather than for minimal prep sets.

Hence, for the most part, min-prep and min-curb satisfy the same natural properties; in their respective axiomatizations, the only distinguishing property concerns the treatment of one-player games. This follows directly from the definitions: prep requires players to hold at least one best response to any belief they may have that is consistent with the recommendations to other players, while curb requires them to hold all best responses. The interesting issue is that while min-prep and min-curb sometimes give rise to very different solutions (see Tercieux and Voorneveld (2005) for some appealing examples), all differences between the two solution concepts are captured by the way they deal with one-player games.

(d) Basu and Weibull (1991) briefly consider socalled minimal curb* sets, a "cautious" variant of minimal curb sets in which players are assumed to abstain from choosing weakly dominated actions.

Formally, let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$, let $i \in N$, and let $a_i \in A_i$. Recall that a_i is *weakly dominated* if there is a mixed strategy $\alpha_i \in \Delta(A_i)$ such that $u_i(a_i, a_{-i}) \le u_i(\alpha_i, a_{-i})$ for each $a_{-i} \in \times_{j\in N\setminus\{i\}} A_j$, with strict inequality for some a_{-i}. The set of actions of player i that are not weakly dominated (sometimes referred to as *admissible*) is denoted by A_i^*.

A *curb* set* of G is a nonempty product set $Q = \times_{i\in N} Q_i \subseteq A$ such that for each $i \in N$ and each belief $\alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(Q_j)$ of player i, the set Q_i contains *all* admissible best responses of player i against his belief:

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(Q_j): \quad BR_i(\alpha_{-i}) \cap A_i^* \subseteq Q_i.$$

A curb* set Q is *minimal* if no curb* set is a proper subset of Q. The set-valued solution concept that assigns to each game its collection of minimal curb* sets is denoted by min-curb*. Hence, for each game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$:

$$\text{min-curb}^*(G) = \{Q \subseteq A \mid Q \text{ is a minimal curb}^* \text{ set of } G\}.$$

It is easily verified that min-curb* satisfies *nonemptiness*, *one-person rationality* as in (7.3), and *nonnestedness*. All other axioms, however, are violated. The main reason for this is that the weak dominance relation may change if one goes from the original game to reduced games or subgames; for instance, an action that is admissible in the original game may be weakly dominated in a reduced game. This indicates intuitively that *consistency* may be violated; we show this formally below and also indicate violations of *satisfaction* and *decisiveness*.

The solution concept min-curb* does not satisfy *consistency*: in the game G in Figure 7.2 we have $Q = \{T, B\} \times \{L\} \in \text{min-curb}^*(G)$. Consider the belief (B, L) in which player 1 chooses B with probability one and player 2 chooses L with probability one. In the reduced game $G^{\{2\},(B,L)} = \langle \{2\}, \{L, C, R\}, v_2\rangle$ with $v_2(L) = v_2(C) = 0, v_2(R) = -1$, action C is no longer weakly dominated and

	L	C	R
T	$1,1$	$1,0$	$0,-1$
B	$1,0$	$0,0$	$1,-1$

(a)

	L	C	R
T	$1,1$	$1,0$	$0,-1$
M	$1,0$	$0,1$	$1,-1$
B	$-1,0$	$-1,1$	$-1,-1$

(b)

Figure 7.2. (a) min-curb* satisfies neither *consistency*, nor *satisfaction*; (b) min-curb* does not satisfy *decisiveness*.

min-curb$^*(G^{\{2\},(B,L)}) = \{\{L, C\}\}$. So $Q_2 = \{L\}$ does not contain a solution of the reduced game $G^{\{2\},(B,L)}$.

The solution concept min-curb* does not satisfy *satisfaction*: in the game G in Figure 7.2(a), $\{T, B\} \times \{L, C\}$ is a satisfactory set under min-curb*, but in the subgame G' restricted to $\{T, B\} \times \{L, C\}$ we have min-curb$^*(G') = \{\{T\} \times \{L\}\} \not\subseteq \{\{T, B\} \times \{L\}\} =$ min-curb$^*(G)$.

The solution concept min-curb* does not satisfy *decisiveness*: in the game G in Figure 7.2(b) we have min-curb$^*(G) = \{\{T, M\} \times \{L, C\}\}$. But in the subgame G' restricted to $\{T, M\} \times \{L, C\}$, M is weakly dominated by T and min-curb$^*(G') = \{\{T\} \times \{L\}\} \neq \{\{T, M\} \times \{L, C\}\}$.

(e) Figure 7.3 summarizes which axioms are satisfied by the key solution concepts in this chapter.

	min-prep	φ_1^s	min-curb (φ_2^s)	φ_3^s	φ_4^s	φ_5^s	min-curb*
nonemptiness	+	−	+	+	+	+	+
one-person rationality	+	+	−	+	+	+	−
consistency	+	+	+	−	+	+	−
nonnestedness	+	+	+	+	−	+	+
satisfaction	+	+	+	+	+	−	−
decisiveness	+	+	+	+	−	+	−
one-person rationality as in (7.3)	−	−	+	−	−	−	+

Figure 7.3. Solution concepts and the axioms they do (+) or do not (−) satisfy.

8 Equilibrium and learning in the minority game

Summary

In this chapter, which is based on Kets and Voorneveld (2007), we characterize the Nash equilibria of a simple congestion game, the minority game, and study the limiting behavior of several well-known learning processes in this game. Interestingly, different learning models yield different predictions for this very simple game. In the next chapter, we discuss an alternative learning model to describe players' behavior in this game, and we compare the predictions of different learning models to experimental results.

8.1 Introduction

Congestion games are ubiquitous in economics. In a congestion game (Rosenthal, 1973), players use several facilities from a common pool. The costs or benefits that a player derives from a facility depends on the number of users of that facility. A congestion game is therefore a natural game to model scarcity of common resources. Examples of such systems include vehicular traffic (Nagel et al., 1997), packet traffic in networks (Huberman and Lukose, 1997), and ecologies of foraging animals (DeAngelis and Gross, 1992). Similar problems are encountered in market entry games (Selten and Güth, 1982).

Congestion games are also interesting from a theoretical point of view. In congestion games, players need to coordinate to differentiate. This seems to be more difficult than coordinating on the same action, as any commonality of expectations is broken up. For instance, when commuters have to choose between two roads A and B and all believe that the others choose A, nobody will choose that road, invalidating beliefs. The sorting of players predicted in the pure-strategy Nash equilibria of such games violates the common belief that in symmetric games, all rational players will evaluate the situation identically, and hence, make the same choices in similar situations (see Harsanyi and Selten, 1988, p. 73). Moreover, in congestion games, players may obtain asymmetric payoffs in equilibrium which may complicate attainment of equilibrium, as coordination cannot be achieved through tacit coordination based on historical precedent (cf. Meyer et al., 1992).

Finally, congestion games often have a large number of equilibria, so that players also face the difficulty of coordinating on the same equilibrium.

Therefore, it is an interesting question what type of behavior different learning models predict in such games. This is the topic of this chapter and the next. We consider a simple congestion game, the minority game. The minority game is based on the El Farol bar problem of Arthur (1994). Players have to choose between two alternatives, where the payoffs of these alternatives only depend on the number of players choosing each option. Congestion is costly, so that players prefer to choose the alternative that is chosen by the smallest number of players. We restrict attention to games with an odd number of players, so that there is always an alternative that is chosen by a smaller number of players than the other, i.e., there is a well-defined minority. In the current chapter, we characterize the equilibria of the game, and study the limiting behavior of a number of well-known learning processes. In the next chapter, we discuss an alternative learning model to describe players' behavior in such games, and compare the different predictions to experimental findings.

The minority game is closely related to the market entry game, a game extensively studied in experimental economics (see the survey of Ochs (1999) and references therein; for a recent contribution see Duffy and Hopkins (2005)). While the market entry game models situations in which players can choose between a safe option (staying out of the market) and an alternative whose payoffs decline in the number of other players choosing that option (entering), the minority game is a suitable model for more symmetric situations in which the payoffs of both actions depend on the number of other players choosing that action. In such situations, players will need to outsmart other players, so as to be one step ahead of their opponents. For instance, the minority game may be a good model for financial markets, where investors try to identify the underpriced shares, and try to sell the shares they expect to fall in the future.

Interestingly, we find that the predictions from different learning processes are not equivocal. While the replicator dynamic predicts that play converges to a Nash equilibrium with at most one player who chooses a mixed strategy, the set of stationary points under the perturbed best-response dynamics consists of the logit quantal response equilibria of the game (McKelvey and Palfrey, 1995); for a definition of these learning processes, see Section 8.3 and 8.4, respectively. For the case of three players, we show that the set of Nash equilibria that are the limit of a sequence of logit quantal response equilibria with vanishing noise consists of the the pure Nash equilibria, the Nash equilibria with one mixer who mixes uniformly, and the Nash equilibrium in which all players randomize equally over their two actions. Finally, we study two best-reply learning process in which players have limited memory: a model proposed by Hurkens (1995) and the model we introduced in Chapter 6. In both these models, players choose an action that is a best reply to some belief over other players' actions that is consistent with their recent past play. The difference between the two models is that in the model of Chapter 6, players also display a so-called recency bias: when there are multiple best replies to a given belief, a player chooses the best reply that he most recently played. We show that while the process of Hurkens offers no sharp predictions for the minority game, the model we introduced in Chapter 6 predicts that play converges to one of the pure Nash equilibria of the game when players have a memory length of at least two periods.

The minority game has been studied by a number of authors in economics. Renault et al. (2005) consider repeated play in the game. Bottazzi and Devetag (2007), Chmura and Pitz (2006), and Helbing et al. (2005) study the game experimentally. The game has also been studied extensively in the physics literature. We relate this literature to the literature on game-theoretic learning models in the next chapter; Challet et al. (2004) or Moro (2003) provide an overview of the minority game literature in physics. Furthermore, the work in this chapter is related to the literature on learning in congestion games and more generally learning in potential games (e.g. Hofbauer and Hopkins, 2005; Hofbauer and Sandholm, 2002; Sandholm, 2001, 2007). Papers that study learning in games similar to the game considered here include Blonski (1999), Franke (2003) and Kojima and Takahashi (2004). Most of these papers focus on the predictions of a single learning model,[1] while we compare predictions from different learning models. Moreover, while most results are obtained for games with either a small number or a continuum of players, we characterize the equilibria of the game and the limiting behavior of different learning processes for any (odd) number of players.

The outline of this chapter is as follows. In Section 8.2, we define the game and characterize its Nash equilibria. In Section 8.3, we characterize the set of stationary states and the set of asymptotically stable states under the replicator dynamic. In Section 8.4, we characterize the set of stationary states under the perturbed best-response dynamics. In Section 8.5, we characterize the limiting behavior in the minority game under the best-reply learning processes with limited memory. Section 8.6 concludes.

[1] Duffy and Hopkins (2005) and Kojima and Takahashi (2004) are notable exceptions.

8.2 The minority game

8.2.1 Basic definitions

Following the notation of Tercieux and Voorneveld (2005), we denote the set of players by $N = \{1, \ldots, 2k + 1\}$, with $k \in \mathbb{N}$. Each player $i \in N$ has a set of pure strategies $A_i = \{-1, +1\}$: agents have to choose between two options. The set of mixed strategies of player i is denoted by $\Delta(A_i)$. We denote a mixed strategy profile by $\alpha \in \times_{i \in N} \Delta(A_i)$, and we use the standard notation $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$ to denote a strategy profile of players other than $i \in N$. A strategy in which a player chooses action $a = -1$ with probability $p \in [0, 1]$ and $a = +1$ with probability $1 - p$ is denoted by $(p, 1 - p)$. With each action $b \in \{-1, +1\}$, a function

$$f_b : \{1, \ldots, 2k + 1\} \to \mathbb{R}$$

can be associated which indicates for each $\ell \in \{1, \ldots, 2k + 1\}$ the payoff $f_b(\ell)$ to a player choosing action b when the total number of players choosing b equals ℓ. The von Neumann-Morgenstern utility function of player $i \in N$ is then given by

$$u_i(a) = f_{a_i}\Big(|\{j \in N \mid a_j = a_i\}|\Big), \tag{8.1}$$

where $a = (a_j)_{j \in N} \in \times_{j \in N} A_j$. Payoffs are extended to mixed strategies in the usual way.

The function (8.1) can have several forms. We make the common assumptions (e.g. Challet et al., 2004) that congestion is costly:

[Mon] f_{-1} and f_{+1} are strictly decreasing functions,

and that the congestion effect is the same across alternatives:

[Sym] $f_{-1} = f_{+1}$.

We refer to a player who uses a mixed strategy that puts positive probability on both pure strategies a *mixer*. A player that puts full probability mass on the alternative -1 is called a (-1)-*player*; similarly, a player that puts full probability mass on the alternative $+1$ is called a $(+1)$-*player*.

8.2.2 Nash equilibria

Throughout this section, let $k \in \mathbb{N}$ and consider a minority game with $2k + 1$ players. We characterize its set of Nash equilibria. The pure Nash equilibria are easy to characterize:

Proposition 8.2.1 (Tercieux and Voorneveld (2005)). *A pure strategy profile is a Nash equilibrium if and only if one of the alternatives* -1 *or* $+1$ *is chosen by exactly* k *of the* $2k+1$ *players.*

It remains to characterize the game's Nash equilibria with at least one mixer.

Lemma 8.2.2. *Let* $\alpha \in \times_{i\in N}\Delta(A_i)$ *be a Nash equilibrium with a nonempty set of mixers. All mixers use the same strategy: for all* $i, j \in N$, *if* $\alpha_i, \alpha_j \notin \{(1,0),(0,1)\}$, *then* $\alpha_i = \alpha_j$.

Proof. By [Sym], the 2×2 subgame played by two mixers i (row player) and j (column player) given the strategy profile of the remaining players is of the form

	-1	$+1$
-1	x, x	y, z
$+1$	z, y	w, w

where, for instance, y is the payoff to the player choosing -1 if the other player chooses $+1$ and the remaining players stick to the mixed strategy profile $(\alpha_k)_{k\in N\setminus\{i,j\}}$. By [Mon], a player is better off if the other chooses differently, i.e., $x < y$ and $z > w$. Let $p, q \in (0,1)$ denote the equilibrium probability with which player i and j, respectively, choose -1. In equilibrium, each player must be indifferent between playing $+1$ and playing -1:

$$px + (1-p)y = pz + (1-p)w,$$
$$qx + (1-q)y = qz + (1-q)w.$$

Subtracting the latter expression from the former yields

$$(p-q)(x-y) = (p-q)(z-w).$$

As $x < y$ and $z > w$, this can only hold if $p = q$. Since mixers i and j were chosen arbitrarily from the set of mixers, this implies that all mixers use the same strategy. □

Since all mixers use the same strategy and player labels are irrelevant by [Sym] (if α is a Nash equilibrium, so is every permutation of α), a non-pure Nash equilibrium can be summarized by its *type* (ℓ, r, λ), where $\ell, r \in \{0, 1, \ldots, 2k+1\}$ denote the number of players choosing pure strategy -1 or $+1$, respectively, and $\lambda \in (0,1)$ the probability with which the remaining $m(\ell, r, \lambda) := (2k+1) - (\ell + r) > 0$ mixers choose -1. Moreover, let $v_{-1}(\ell, r, \lambda)$ denote the expected payoff to a player choosing -1; $v_{+1}(\ell, r, \lambda)$ is defined similarly. For convenience, write $m := m(\ell, r, \lambda)$.

Letting one of the mixers in (ℓ, r, λ) deviate to a pure strategy, this implies in particular that

$$v_{-1}(\ell+1, r, \lambda) = \sum_{s=0}^{m-1} \binom{m-1}{s} \lambda^s (1-\lambda)^{m-1-s} f_{-1}(\ell+1+s), \tag{8.2}$$

$$\begin{aligned} v_{+1}(\ell, r+1, \lambda) &= \sum_{s=0}^{m-1} \binom{m-1}{s} \lambda^s (1-\lambda)^{m-1-s} f_{+1}((r+1)+(m-1-s)) \\ &= \sum_{s=0}^{m-1} \binom{m-1}{s} \lambda^s (1-\lambda)^{m-1-s} f_{+1}(r+m-s). \end{aligned} \tag{8.3}$$

For instance, a profile of type $(\ell+1, r, \lambda)$ is obtained from type (ℓ, r, λ) if a mixer switches to pure strategy -1. In that case, there are $m-1$ mixers left. To obtain expected payoffs, notice that the probability that $s \in \{0, \ldots, m-1\}$ of these mixers choose -1 is $\binom{m-1}{s} \lambda^s (1-\lambda)^{m-1-s}$. Using this notation, the Nash equilibria with at least one mixer are characterized as follows:

Proposition 8.2.3.

(a) (Characterization of equilibrium) *Let $\ell, r \in \{0, 1, \ldots, 2k+1\}$ be such that $\ell + r < 2k+1$. Let $\lambda \in (0, 1)$. A strategy profile of type (ℓ, r, λ) is a Nash equilibrium if and only if*

$$v_{-1}(\ell+1, r, \lambda) = v_{+1}(\ell, r+1, \lambda). \tag{8.4}$$

(b) (Equilibria with one mixer) *There exist equilibria with exactly one mixer. These equilibria are of type (k, k, λ) with arbitrary $\lambda \in (0, 1)$, i.e., the mixer uses an arbitrary mixed strategy, whereas the remaining $2k$ players are spread evenly over the two pure strategies.*

(c) (Equilibria with more than one mixer) *Let $\ell, r \in \{0, 1, \ldots, 2k+1\}$ be such that $\ell + r \leq 2k-1$. There is a Nash equilibrium of type (ℓ, r, λ) if and only if $\max\{\ell, r\} < k$. The corresponding probability $\lambda \in (0, 1)$ solving (8.4) is unique.*

Proof. (a): Condition (8.4) says that a mixer is indifferent between choosing -1, thereby raising ℓ to $\ell+1$ and obtaining payoff $v_{-1}(\ell+1, r, \lambda)$, or choosing $+1$, thereby raising r to $r+1$ and obtaining payoff $v_{+1}(\ell, r+1, \lambda)$. Hence, (8.4) is a necessary condition for Nash equilibrium.

To establish sufficiency, it remains to show that also players using a pure strategy—if there are such players, i.e., if $\ell + r \geq 1$—choose a best reply. Suppose

$\ell \geq 1$. The payoff to a (-1)-player is $v_{-1}(\ell, r, \lambda)$, while a unilateral deviation to $+1$ yields $v_{+1}(\ell - 1, r + 1, \lambda)$. However:

$$\begin{aligned} v_{-1}(\ell, r, \lambda) &\geq v_{-1}(\ell + 1, r, \lambda) && (8.5)\\ &= v_{+1}(\ell, r + 1, \lambda) && (8.6)\\ &\geq v_{+1}(\ell - 1, r + 1, \lambda). && (8.7) \end{aligned}$$

Inequality (8.5) uses [Mon]: conditioning on the behavior of one of the $m := m(\ell, r, \lambda) > 0$ mixers, write

$$v_{-1}(\ell, r, \lambda) = \lambda v_{-1}(\ell + 1, r, \lambda) + (1 - \lambda) v_{-1}(\ell, r + 1, \lambda).$$

Then

$$\begin{aligned} &v_{-1}(\ell, r, \lambda) - v_{-1}(\ell + 1, r, \lambda) = (1 - \lambda)\left[v_{-1}(\ell, r + 1, \lambda) - v_{-1}(\ell + 1, r, \lambda)\right]\\ &= (1 - \lambda) \sum_{s=0}^{m-1} \binom{m-1}{s} \lambda^s (1 - \lambda)^{m-1-s} \left[f_{-1}(\ell + s) - f_{-1}(\ell + 1 + s)\right]\\ &\geq 0 \end{aligned}$$

by [Mon]. Inequality (8.7) follows similarly and (8.6) is simply condition (8.4). So if $\ell \geq 1$, (-1)-players choose a best reply. Similarly, if $r \geq 1$, $(+1)$-players choose a best reply.

(b): Let $\lambda \in (0, 1)$. Substitution in (8.4) and [Sym] yield that strategy profiles of type (k, k, λ) are Nash equilibria:

$$v_{-1}(k + 1, k, \lambda) = f_{-1}(k + 1) = f_{+1}(k + 1) = v_{+1}(k, k + 1, \lambda).$$

Conversely, consider a Nash equilibrium of type (ℓ, r, λ) with exactly one mixer: $\ell + r = 2k$. We establish that $\ell = r$. Suppose not. Without loss of generality, $\ell > r$. Since $\ell + r = 2k$, this implies $\ell \geq k + 1$ and $r \leq k - 1$. The expected payoff to a (-1)-player is

$$\lambda f_{-1}(\ell + 1) + (1 - \lambda) f_{-1}(\ell),$$

while deviating to $+1$ would yield

$$\lambda f_{+1}(r + 1) + (1 - \lambda) f_{+1}(r + 2).$$

Since $\ell + 1 > r + 1$, $\ell \geq r + 2$, and $\lambda \in (0, 1)$, it follows from [Sym] and [Mon] that a (-1)-player would benefit from unilateral deviation, contradicting the assumption that the profile of type (ℓ, r, λ) is a Nash equilibrium. Conclude that $\ell = r$.

(c): Without loss of generality, $\ell \geq r$, so $\max\{\ell, r\} = \ell$. Let $m = (2k+1) - (\ell + r) \geq 2$ be the number of mixers. By substitution, $\ell < k$ if and only if $\ell + 1 < r + m$. To prove (c), it therefore remains to establish three things.

Firstly, if $\ell + 1 < r + m$, there is a $\lambda \in (0,1)$ solving (8.4). To see this, use $\ell \geq r$ to find that $\ell + m > r + 1$. By [Sym] and [Mon], it follows that

$$\begin{array}{rcccccl} v_{-1}(\ell+1, r, 0) & = & f_{-1}(\ell+1) & > & f_{+1}(r+m) & = & v_{+1}(\ell, r+1, 0), \\ v_{-1}(\ell+1, r, 1) & = & f_{-1}(\ell+m) & < & f_{+1}(r+1) & = & v_{+1}(\ell, r+1, 1). \end{array}$$

By the Intermediate Value Theorem applied to $v_{-1}(\ell+1, r, \cdot) - v_{+1}(\ell, r+1, \cdot)$, there is a $\lambda \in (0,1)$ solving (8.4): there is a Nash equilibrium of type (ℓ, r, λ).

Secondly, this $\lambda \in (0,1)$ solving (8.4) is unique. By (8.2), $v_{-1}(\ell+1, r, \cdot)$ is the expectation of a strictly decreasing function of a binomial stochastic variable. By stochastic dominance (see Appendix 8.A), this makes $v_{-1}(\ell+1, r, \cdot)$, the left-hand side of (8.4), strictly decreasing in λ. Similarly, by (8.3), the right-hand side of (8.4) is strictly increasing in λ. Conclude that the functions $v_{-1}(\ell+1, r, \cdot)$ and $v_{+1}(\ell, r+1, \cdot)$ intersect at most once. By the previous step, as long as $\ell + 1 < r + m$, they intersect at least once, establishing uniqueness.

Thirdly, if $\ell + 1 \geq r + m$, there is no $\lambda \in (0,1)$ solving (8.4). To see this, notice that the inequality implies

$$\ell + m > \cdots > \ell + 2 > \ell + 1 \geq r + m > r + m - 1 > \cdots > r + 1,$$

so by [Sym] and [Mon]:

$$f_{-1}(\ell+m) < \cdots < f_{-1}(\ell+2) < f_{-1}(\ell+1) \leq f_{+1}(r+m) < f_{+1}(r+m-1) < \cdots < f_{+1}(r+1).$$

Substitution in (8.2) and (8.3) yields that

$$v_{+1}(\ell, r+1, \lambda) > v_{-1}(\ell+1, r, \lambda)$$

for all $\lambda \in (0,1)$: there is no solution to (8.4). □

Implications of this characterization of the game's non-pure Nash equilibria include:

(*i*) There are no Nash equilibria where the number of mixers is two, since in that case, $\max\{\ell, r\} \geq k$.

(*ii*) Substitution in (8.4) gives that a strategy profile in which the number of (-1)-players is equal to the number of $(+1)$-players and the remaining players mix with probability $1/2$, i.e., a profile of type $(t, t, 1/2)$ with $t \in \{0, \ldots, k\}$, is a Nash equilibrium.

Having characterized the set of Nash equilibria, we now establish that the set of Nash equilibria with at most one mixer is connected.

Proposition 8.2.4. *The set of Nash equilibria with at most one mixer is connected.*

Proof. In a Nash equilibrium with exactly one mixer, the completely mixed strategy is arbitrary. Letting the probability go to zero or one, this line piece of Nash equilibria in the strategy space has a pure Nash equilibrium as its end point. Hence, to show connectedness, it suffices to show that for each pair of pure Nash equilibria, there is a chain of pure Nash equilibria differing in exactly one coordinate connecting them.

So let x and y be distinct pure Nash equilibria. By Proposition 8.2.1, the majority action, i.e., the action chosen by exactly $k+1$ players in a given Nash equilibrium, is well defined. We need to consider two cases. Firstly, if this action is the same in x and y, say -1, then $x \neq y$ implies that the $(k+1)$-player majorities in x and y must be distinct. Let i be such a majority player, choosing -1 in x, but $+1$ in y. Secondly, if the majority action is different in x and y, say -1 in x and $+1$ in y, then by definition of a majority, the $(k+1)$-player majorities in x and y have a nonempty intersection. Again, let i be a majority player choosing -1 in x, but $+1$ in y.

By construction, as i is a majority player, the path of Nash equilibria in which i increases the probability of playing the action $+1$ from 0 to 1 connects x to another pure Nash equilibrium x^* with $x_i \neq x_i^* = y_i$ and $x_j^* = y_j$ for all $j \neq i$, i.e., with a strictly smaller Hamming distance to y (recall that the Hamming distance between two finite-dimensional vectors is the number of coordinates in which they differ).

As the strategy vectors only have a finite number of coordinates and we can reduce the Hamming distance between pure Nash equilibria by the procedure above, the result now follows by induction. □

8.3 The replicator dynamic

In this section, we study the (multipopulation) replicator dynamic (e.g. Weibull, 1995) for the minority game. There is a set $N = \{1, \ldots, 2k+1\}$ of populations, where each population is the unit interval $[0, 1]$. The populations represent the $2k+1$ player positions in the minority game. All agents in a population are initially programmed to some pure strategy. Hence, each population can be divided into two subpopulations (one of which may contain no agents), one for each of the pure strategies in the minority game. A *population state* is a vector $\alpha = (\alpha_1, \ldots, \alpha_{2k+1})$ in the polyhedron of mixed-strategy profiles, where for each $i \in N$, α_i is a point in the simplex $\Delta(A_i)$, representing the distribution of agents in population i across the different pure strategies. The vector $\alpha_i \in \Delta(A_i)$ thus represents the state of population i, with $\alpha_i(a_i)$ denoting the proportion of agents programmed to play the pure strategy $a_i \in A_i$.

Time is continuous and indexed by t. Agents, one from each population, are continuously drawn uniformly at random from these populations to play the minority game. Suppose payoffs represent the effect of playing the game on an agent's fitness, measured as the number of offspring per time unit, and that each offspring inherits its single parent's strategy. This gives rise to the following dynamics for the population shares:

$$\forall i \in N, \forall a_i \in A_i : \quad \dot{\alpha}_i(a_i) = \alpha_i(a_i)(u_i(a_i, \alpha_{-i}) - u_i(\alpha_i, \alpha_{-i})). \tag{8.8}$$

This system of differential equations defines the *(continuous time multipopulation) replicator dynamic*. In words, the growth rate $\dot{\alpha}_i(a_i)/\alpha_i(a_i)$ of a pure strategy $a_i \in A_i$ in population $i \in N$ is equal to the difference in payoffs of the pure strategy and the current average payoffs for the population. Hence, the population shares of strategies that do better than average will grow, while the shares of the other strategies will decline. It is easily seen that the subpopulations associated with the pure best replies to the current population state have the highest growth rates.

The system of differential equations (8.8) defines a continuous *solution mapping* $\xi : \mathbb{R} \times \left(\times_{i \in N} \Delta(A_i)\right) \to \times_{i \in N} \Delta(A_i)$ which assigns to each time $t \in \mathbb{R}$ and each initial state $\alpha^0 \in \times_{i \in N} \Delta(A_i)$ the population state $\xi(t, \alpha^0) \in \times_{i \in N} \Delta(A_i)$. The (solution) *trajectory* through a population state $\alpha^0 \in \times_{i \in N} \Delta(A_i)$ is the graph of the solution mapping $\xi(\cdot, \alpha^0)$.

A population state $\alpha \in \times_{i \in N} \Delta(A_i)$ is a *stationary state* of the replicator dynamics (8.8) if and only if for each population $i \in N$ all pure strategies $a_i \in A_i$ that are used by some agents in the population give the same payoffs. In that case, $\dot{\alpha}_i(a_i) = 0$ for all $i \in N$ and all $a_i \in A_i$. Let

$$S := \left\{\alpha \in \times_{j \in N} \Delta(A_j) \;\middle|\; \forall i \in N, \forall a_i \in A_i : \dot{\alpha}_i(a_i) = 0\right\}$$

be the set of stationary states. By definition, if $\alpha \in S$, then a player $i \in N$ either uses a pure strategy or—if he is a mixer—is indifferent between his two pure strategies:

$u_i(a_i, \alpha_{-i}) = u_i(\alpha_i, \alpha_{-i})$ for both $a_i \in A_i$. Using the proof of Lemma 8.2.2, all mixers must use the same strategy. If there is more than one mixer, the proof of Proposition 8.2.3(c) indicates that this mixed strategy solving (8.4) is uniquely determined by the number of players choosing pure strategy -1 and pure strategy $+1$. Conclude that the set of stationary states can be partitioned into three subsets:

S_1: The connected set of Nash equilibria with at most one mixer;

and a finite collection of isolated stationary states, namely

S_2: Nash equilibria with more than one mixer;

S_3: nonequilibrium profiles of some type (ℓ, r, λ), where

$$\begin{cases} \ell, r \in \{0, \dots, 2k+1\}, \\ \ell + r \leq 2k+1, \\ \text{if } \ell + r < 2k+1, \text{ then } \lambda \in (0,1) \text{ uniquely determined by (8.4).} \end{cases}$$

It remains to study the stability properties of these stationary states. We consider Lyapunov stability and asymptotic stability. Roughly speaking, a population state is Lyapunov stable if no small change in the population shares can lead the replicator dynamics away from the population state, while a population state is asymptotically stable if it is Lyapunov stable and in addition any sufficiently small change in the population shares results in a movement back to the original population state. Formally, a population state $\alpha \in \times_{i \in N} \Delta(A_i)$ is *Lyapunov stable* if every neighborhood B of α contains a neighborhood B^0 of α such that $\xi(t, \alpha^0) \in B$ for every $x^0 \in B \cap \times_{i \in N} \Delta(A_i)$ and $t \geq 0$. It is *asymptotically stable* if it is Lyapunov stable, and, in addition, there exists a neighborhood B^* such that

$$\lim_{t \to \infty} \xi(t, \alpha^0) = \alpha$$

for each initial state $\alpha^0 \in B^* \cap \times_{i \in N} \Delta(A_i)$.

The analysis relies heavily on the existence of a Lyapunov function for the replicator dynamic in the minority game. Tercieux and Voorneveld (2005), using Theorem 3.1 in Monderer and Shapley (1996), show that a minority game is a (finite exact) potential game. That is, there exists a real-valued (so-called potential) function U on the pure strategy space such that for each $i \in N$, each $a_{-i} \in \times_{j \in N \setminus \{i\}} A_j$, and all $a_i, b_i \in A_i$:

$$u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) = U(a_i, a_{-i}) - U(b_i, a_{-i}). \tag{8.9}$$

Taking expectations, (8.9) can be extended to mixed strategies, so the payoff difference in (8.8) equals the corresponding change in the potential. Hence, the replicator dynamic can be rewritten as:

$$\forall i \in N, \forall a_i \in A_i : \quad \dot{\alpha}_i(a_i) = \alpha_i(a_i)(U(a_i, \alpha_{-i}) - U(\alpha_i, \alpha_{-i})). \tag{8.10}$$

This makes the potential U a Lyapunov function of the replicator dynamic. More precisely:

Proposition 8.3.1. *The potential function U of the minority game is a strict Lyapunov function for the replicator dynamic: for each solution trajectory $(\alpha(t))_{t\in[0,\infty)}$, $dU(\alpha(t))/dt \geq 0$ with equality exactly in the stationary states.*

Proof. For $i \in N$, we write $\mathbb{E}_{\alpha_i}$ and Var_{α_i} to denote the expectation and variance with respect to $\alpha_i \in \Delta(A_i)$, respectively. Suppressing time indices for ease of notation, direct calculation gives

$$\begin{aligned}
\dot{U}(\alpha) &= \sum_{i\in N} \sum_{a_i\in A_i} U(a_i, \alpha_{-i})\dot{\alpha}_i(a_i) \\
&= \sum_{i\in N} \sum_{a_i\in A_i} \alpha_i(a_i)(U(a_i, \alpha_{-i}) - U(\alpha_i, \alpha_{-i}))U(a_i, \alpha_{-i}) \\
&= \sum_{i\in N} \sum_{a_i\in A_i} \left(\alpha_i(a_i)U(a_i, \alpha_{-i})^2 - U(\alpha_i, \alpha_{-i})^2\right) \\
&= \sum_{i\in N} \left(\mathbb{E}_{\alpha_i}\left[U(a_i, \alpha_{-i})^2\right] - \left(\mathbb{E}_{\alpha_i}\left[U(a_i, \alpha_{-i})\right]\right)^2\right) \\
&= \sum_{i\in N} \mathrm{Var}_{\alpha_i}\left[U(a_i, \alpha_{-i})\right] \\
&\geq 0,
\end{aligned}$$

with equality if and only if all variances are zero, i.e., if and only if α is a stationary point of the replicator dynamics. □

Proposition 8.3.2. *The collection of Nash equilibria with at most one mixer in S_1 is asymptotically stable under the replicator dynamic. Stationary states in S_2 and S_3 are not Lyapunov stable.*

Proof. To see that the collection of Nash equilibria in S_1 is asymptotically stable, notice that S_1 is the set of global maxima of U: the potential U in (8.9) has been extended to mixed strategies by taking expectations, so U achieves a global maximum in a pure strategy profile which, again by (8.9), is a pure Nash equilibrium. By symmetry, all pure Nash equilibria are global maxima of U and so are equilibria with exactly one mixer. Other strategy profiles are not global maxima of U: they are not Nash equilibria or, if they are, they involve more than one mixer, in which case they put positive probability also on pure strategy profiles that are not Nash equilibria and consequently not global maxima of U. This connected set of global maxima of the Lyapunov function U is asymptotically stable (Weibull, 1995, Thm. 6.4).

We show that elements of S_2 are not Lyapunov stable; the case for points in S_3 is similar. Let $\alpha^* \in S_2$, i.e, α^* is a Nash equilibrium with more than one mixer. Suppose it is Lyapunov stable. Since it is an isolated point of the collection of stationary states, there is a neighborhood B of α^* whose closure contains only the stationary state α^*: $\text{cl}(B) \cap S_2 = \{\alpha^*\}$. By Lyapunov stability, as long as the initial state $\alpha(0)$ lies in a sufficiently small neighborhood B' of α^*, the entire solution trajectory $(\alpha(t))_{t\in[0,\infty)}$ remains in B.

Let $i \in N$ be one of the mixers in the Nash equilibrium α^*. Since i is indifferent between his two pure strategies and the potential U measures payoff differences, it follows that

$$U(\alpha^*) = U(-1, \alpha^*_{-i}) = U(+1, \alpha^*_{-i}).$$

Consequently, $U(\gamma_i, \alpha^*_{-i}) = U(\alpha^*)$ for all mixed strategies γ_i of player i. For $\gamma_i \neq \alpha^*_i$ sufficiently close to α^*_i, it follows that $(\gamma_i, \alpha^*_{-i}) \in B'$. Hence, the entire solution trajectory $(\gamma(t))_{t\in[0,\infty)}$ with $\gamma(0) := (\gamma_i, \alpha^*_{-i})$ remains in B. Since its starting point is not stationary, Proposition 8.3.1 implies that the Lyapunov function U strictly increases along the trajectory, until it may reach a stationary state. Let $\gamma^* \in \times_{j\in N}\Delta(A_j)$ be a limit point of the trajectory $(\gamma(t))_{t\in[0,\infty)}$: there is a strictly increasing sequence of time points $t_m \to \infty$ such that $\lim_{m\to\infty} \gamma(t_m) \to \gamma^*$. Such a limit point exists and has to be a stationary point (Lemma A.1 of Sandholm, 2001, p. 104). Since $\text{cl}(B) \cap S_2 = \{\alpha^*\}$ and the trajectory lies in B, it follows that $\gamma^* = \alpha^*$. But then $\lim_{m\to\infty} U(\gamma(t_m)) = U(\alpha^*) = U(\gamma(0))$, contradicting that the Lyapunov function is increasing along the trajectory. Conclude that α^* cannot be Lyapunov stable. For $\alpha^* \in S_3$, proceed similarly. As it is not a NE, some i can profitably deviate slightly (to remain inside B'), so the remaining trajectory must increase the potential, but still have α^* as its limit point. □

8.4 Perturbed best-response dynamics and quantal response equilibria

8.4.1 Perturbed best-response dynamics

Under stochastic fictitious play (e.g. Hofbauer and Hopkins, 2005; Hofbauer and Sandholm, 2002; Hopkins, 2002), players repeatedly play a normal form game (in discrete time). They choose best replies to their beliefs on other players' actions on the basis of a perturbed payoff function, with beliefs determined by the time average of past play. More specifically, the state variable at time $t \in \mathbb{N}$ is a vector $Z^t \in \times_{i\in N}\Delta(A_i)$, where the ith component Z_i^t denotes the time average of player i's past play up to time t. Players' initial choices are arbitrary pure strategies; in later periods players best-respond to their beliefs Z^t, after their payoffs have been subjected to random shocks. That is, for each $i \in N$, let $(\varepsilon_i^a)_{a\in A_i}$ be a vector of payoff disturbances. The vector of payoff disturbances is independent and identically distributed across players and over time. Let $\alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(A_j)$ be a belief. The probability that player i chooses action $a_i \in A_i$ is equal to the probability that

$$u_i(a_i, \alpha_{-i}) + \varepsilon_i^{a_i} \geq u_i(b_i, \alpha_{-i}) + \varepsilon_i^{b_i}$$

for all $b_i \in A_i$. Then, the perturbed best-response dynamics associated with Gumbel-distributed perturbations with parameter $\beta > 0$ is:

$$\forall i \in N, \forall a_i \in A_i : \quad \dot{\alpha}_i(a_i) = \frac{\exp\left[\beta u_i(a_i, \alpha_{-i})\right]}{\sum_{b_i\in A_i} \exp\left[\beta u_i(b_i, \alpha_{-i})\right]} - \alpha_i(a_i). \tag{8.11}$$

Gumbel-distributed payoff perturbations correspond to control costs of the relative entropy form. By Proposition 4.1 of Hofbauer and Sandholm (2002), the process in (8.11) has a strict Lyapunov function that can be expressed in terms of the potential function and the control cost functions. For each $i \in N$, let α_i denote the probability with which player i chooses the action $a_i = -1$. Then, the Lyapunov function for the process in (8.11) is defined by:

$$\forall \alpha \in \times_{i\in N}\Delta(A_i) : \quad V(\alpha) = U(\alpha) - \frac{1}{\beta}\sum_{i\in N}\left[\alpha_i \log(\alpha_i) + (1-\alpha_i)\log(1-\alpha_i)\right], \tag{8.12}$$

where U is the potential function. Since control cost functions of the relative entropy form satisfy the smoothness conditions of Proposition 4.2 of Hofbauer and Sandholm (2002), it follows that:

Proposition 8.4.1. *The collection of stationary states and recurrent points of the process in* (8.11) *coincide.*

Theorem 6.1(iii) of Hofbauer and Sandholm (2002) now implies that the perturbed best-response dynamic converges to these stationary states. Notice that the set of stationary states coincides with the set of logit quantal response equilibria of the minority game (McKelvey and Palfrey, 1995). When the perturbation terms go to zero, we obtain Nash equilibria. As the set of Nash equilibria is not finite, we cannot apply Corollary 6.6 of Benaïm (1999) to characterize the subset of Nash equilibria

	-1	$+1$
-1	-1	0
$+1$	0	0

	-1	$+1$
-1	0	0
$+1$	0	-1

Figure 8.1. A potential function of the 3-player minority game

to which the stochastic process (8.11) converges. The set of Nash equilibria that are the limit points of a sequence of logit quantal response equilibria is generally hard to characterize. In the next section, we characterize this set for the three-player minority game.

8.4.2 Stationary points for the three-player minority game

Consider the three-player minority game with $f_{-1} = f_{+1} = f$ strictly decreasing in the number of users. As it involves a simple rescaling of functions satisfying [Mon] and [Sym], we may without loss of generality set $f(2) = 0$ and $f(1) - f(3) = 1$. A potential of the game is then given in Figure 8.1. The Nash equilibria of the three-player game follow easily from the results in Section 8.2.2. Throughout this section, Nash equilibria are denoted by $(p, q, r) \in [0, 1]^3$, where p, q, r are the probabilities with which player 1, 2, and 3, respectively, choose -1. Then, the Nash equilibria of the game are $(1/2, 1/2, 1/2)$ and $(1, 0, \lambda)$ for some $\lambda \in [0, 1]$, and permutations of these.

Given parameter $\beta \geq 0$, the conditions for a logit quantal response equilibrium (QRE) become:

$$p = \frac{1}{1 + \exp\left[-\beta(1 - q - r)\right]}, \tag{8.13}$$

$$q = \frac{1}{1 + \exp\left[-\beta(1 - p - r)\right]}, \tag{8.14}$$

$$r = \frac{1}{1 + \exp\left[-\beta(1 - p - q)\right]}. \tag{8.15}$$

Given $\beta \geq 0$, we denote a logit QRE in which player 1,2 and 3 play -1 with probability p, q, r by (p, q, r, β). We now characterize the set of Nash equilibria that are the limit of a sequence of quantal response equilibria when $\beta \to \infty$.

Proposition 8.4.2. *Let $(p(\beta_n), q(\beta_n), r(\beta_n), \beta_n)_{n \in \mathbb{N}}$ be a sequence of logit quantal response equilibria: $\beta_n \to \infty$ and for each $n \in \mathbb{N}$, the quadruple $(p(\beta_n), q(\beta_n), r(\beta_n), \beta_n)$ solves* *equations* (8.13)-(8.15). *A Nash equilibrium* (p, q, r) *is the limit of such a sequence if and only if one of the following conditions hold:*

(a) (p, q, r) *is a pure Nash equilibrium,*
(b) (p, q, r) *is a Nash equilibrium with exactly one mixer who mixes uniformly,*
(c) $(p, q, r) = (1/2, 1/2, 1/2)$.

The proof is in Appendix 8.B. Proposition 8.4.2 thus characterizes the set of stationary points of the perturbed best response dynamics (8.11) for the three-player minority game.

8.5 Best-reply learning with limited memory

In this section, we consider discrete time learning models in which players choose best replies to beliefs that are supported by observed play in the recent past. We study two such models, the learning model proposed by Hurkens (1995) and the model proposed in Chapter 6.

In the learning model of Hurkens, players may choose any action that is a best reply to some belief over other players' actions that is consistent with their recent past play. The limiting behavior of this learning process is easy to characterize. Hurkens shows that the Markov processes defined by his learning process eventually settle down in minimal curb sets (Basu and Weibull, 1991). Recall (Section 2.1.2) that a curb set is a product set of pure strategies that contains *all* best replies against beliefs consistent with the recommendations of the other players; a curb set is minimal if it does not contain another curb set. Unfortunately, the learning process of Hurkens does not provide a sharp prediction in the minority game. As shown by Tercieux and Voorneveld (2005), the unique minimal curb set in the minority game consists of the entire strategy space. That is, over time, all players will continue to choose both actions.

By contrast, the learning model introduced in Chapter 6, in which players best-reply to their beliefs subject to a so-called recency bias, offers sharp predictions. As shown in Chapter 6, play converges to one of the minimal prep sets of the game under this learning process. Recall (Section 2.1.2) that a prep set is a product set of pure strategies that contains *at least one* best reply against beliefs consistent with the recommendations of the other players; a prep set is minimal if it does not contain another prep set. Tercieux and Voorneveld (2005) show that the minimal prep sets of the minority game and the pure Nash equilibria of the game coincide. Hence, under the learning model of Chapter 6, play in the minority game converges to one of the pure Nash equilibria of the game.

In both learning models, players need to recall a sufficiently long period of play in order for play to converge. We now turn to the question of the lower bound on players' memory. In Chapter 6, we have stated a sufficient condition on players' memory length for play to converge for any finite strategic game, but we also remarked that this bound may not be tight. By exploiting specific features of a game, the bound can be relaxed for certain games. Hence, an interesting question is whether we can relax the lower bound given in Chapter 6 for minority games.

Suppose players remember actions that were chosen during the past $T \in \mathbb{N}$ periods. It is easy to see that a memory length of $T = 1$ is insufficient for a best-reply learning process with limited memory to converge in the minority game (for generic initial conditions). Suppose that at time t, players choose an action profile that is not a pure Nash equilibrium of the game. Then, there is an action, say -1, that is chosen by more than $k + 1$ players. Hence, at time $t + 1$, all players chooses the unique best reply $+1$. Repeating this argument, we see that play will cycle forever between action profiles in which all players choose -1 and action profiles in which all players choose $+1$.

However, we show that a memory length of $T = 2$ is sufficient for the minority game for play to converge to one of the pure Nash equilibria under the learning model of Chapter 6. In showing this, we give a simple proof of the general convergence result in Chapter 6 (Theorem 6.4.1) for the special case of the minority game, which may provide the reader with helpful intuition for the general case.

For ease of reference, we recall the most important definitions from Chapter 6 here. When the memory length T is equal to 2, the process is a Markov chain with state space

$$H = \{(a^1, a^2) \mid a^1, a^2 \in \times_{j \in N} A_j\}.$$

A history $h = (a^1, a^2) \in H$ indicates that players chose the action profiles a^1 and a^2 one and two periods ago, respectively. The transition probability functions $P : H \times H \to [0, 1]$ give the probability of moving from one state to the next, i.e., $P(h, h') \in [0, 1]$ is the probability of moving from state $h \in H$ to state $h' \in H$. By definition, $\sum_{h' \in H} P(h, h') = 1$ for all $h \in H$. It is not necessary to specify the relevant probabilities: for the convergence result, only sign restrictions are needed.

Recall that the class of learning processes studied in Chapter 6 satisfies two conditions. If $P(h, h') > 0$, then histories $h, h' \in H$ are such that:

P1: The history $h' = (b^1, b^2)$ is a successor of $h = (a^1, a^2)$, i.e., $b^2 = a^1$.
P2: For each $i \in N$, b_i^1 is a best reply to some belief $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(\{a_j^1, a_j^2\})$. Moreover, $b_i^1 = a_i^1$ if and only if a_i^1 is a best reply to α_{-i}.

That is, going from h to h', a new profile of most recent actions is appended to h (P1), and each player chooses a best reply to a belief with support in the product set of actions chosen in the previous $T = 2$ periods, selecting the most recent best reply if such a best reply exists (P2). In games with two actions such as the minority game, this condition simply means that players continue to play the action they chose in the previous round, unless it is no longer a best reply to his current belief.

Proposition 8.5.1. *Consider a Markov chain on H with transition probability function P, where, for all states $h, h' \in H$, it holds that $P(h, h') > 0$ if and only if [P1] and [P2] are satisfied. This Markov process eventually settles down in a pure Nash equilibrium.*

Proof. Let $h_0 = (a^1, a^2) \in H$ and distinguish two cases:

Case 1: a^1 is a pure Nash equilibrium. By [P2], with positive probability, each player chooses his action under the belief that all his opponents will play according to a^1 in the next period. Each player's most recent best reply is to continue playing as in a^1, so the process moves with positive probability to the history $h_1 = (a^1, a^1)$. From this state onwards, the only feasible belief based on the past two periods is that all players play according to a^1 and the most recent best reply implies that they will continue to play according to a^1: the process remains in state h_1 and play has converged to a pure Nash equilibrium.

Case 2: a^1 is not a pure Nash equilibrium. By Proposition 8.2.1, one of the actions, without loss of generality -1, was chosen by a set $S \subseteq N$ of players with $|S| > k + 1$. Each player's unique best response to a^1 is therefore to choose $+1$. By [P2], the process moves with positive probability to state $h_1 = ((+1, \ldots, +1), a^1)$. Let $a^* \in \times_{j \in N} A_j$ be a pure Nash equilibrium in which $k + 1$ members of S choose $+1$ and the others choose -1. Again using [P2], the process moves with positive probability from h_1 to $h_2 = (a^*, (+1, \ldots, +1))$:

- For each of the selected $k + 1$ members of S, $+1$ is the unique best reply to the belief drawn from the past two periods that at least $k + 1$ other players from S will choose -1.
- For each of the remaining k players, -1 is the unique best response to the belief that all other players will continue to play last period's profile $(+1, \ldots, +1)$.

Notice that history h_2 belongs to case 1.

Conclude that, regardless of the initial state h_0, the Markov process moves with positive probability to an absorbing state where the players continue to play one of the game's pure Nash equilibria. As the Markov process is finite and the initial state was chosen arbitrarily, this will eventually happen with probability one (Kemeny and Snell, 1976): play eventually settles down in a pure Nash equilibrium. □

Remark 8.5.2. Notice that, due to the symmetry of the minority game, by slightly adapting the proof it can be shown that play converges to one of the pure Nash equilibria of the game if players only remember their own actions in the past two periods, as well as *how many others* chose these actions. This comes at the expense of a more complex notation and a larger deviation from the notation of Chapter 6. ◁

8.6 Conclusions

Though congestion games are apparently simple, game-theorists' understanding of play in such games is far from complete, for two reasons. Firstly, well-known learning models do not always provide equivocal predictions for such games. In the current chapter, we have characterized the Nash equilibria and the limiting behavior of several well-known learning models in a simple congestion game, the minority game. We have shown that these learning models provide different predictions for this game. Secondly, experimental results are not always in line with theoretical predictions. In experiments on market entry games, aggregate play is largely consistent with equilibrium play, with the number of entrants close to the market capacity. However, individual play generally does not resemble Nash play (see e.g. Ochs, 1999). In the next chapter, we therefore discuss an alternative learning model which seems especially suitable to describe players' behavior in such games, and compare the predictions of this learning model to the predictions of the learning models discussed in the current chapter, and to experimental results.

8.A Stochastic dominance for binomial distributions

Let X have a binomial distribution with $n \in \mathbb{N}$ draws and success probability $p \in [0, 1]$; briefly, a $B(n, p)$ distribution: $X = X_1 + \cdots + X_n$, where $X_1, \ldots, X_n$ are i.i.d $B(1, p)$. Distributions with a higher success rate p *stochastically dominate* those with a lower one (cf. Ross, 1996, Exc. 9.9). Formally, in terms of cumulative distributions, if $p, q \in [0,1]$ and $p < q$, then

$$\text{for all } m \in \{0, \ldots, n\}: \quad \sum_{k=0}^{m} \binom{n}{k} p^k (1-p)^{n-k} \geq \sum_{k=0}^{m} \binom{n}{k} q^k (1-q)^{n-k},$$

with strict inequality if $m < n$. This follows by substitution if $m = 0$ or $m = n$. So let $m \in \{1, \ldots, n-1\}$. It suffices to show that the function

$$[0,1] \ni p \mapsto \sum_{k=0}^{m} \binom{n}{k} p^k (1-p)^{n-k}$$

has a negative derivative on $(0,1)$. The derivative, after rewriting, becomes

$$\begin{aligned}
&\sum_{k=0}^{m} \binom{n}{k} \left[k p^{k-1} (1-p)^{n-k} - (n-k) p^k (1-p)^{n-k-1} \right] \\
&= \sum_{k=0}^{m} \binom{n}{k} p^{k-1} (1-p)^{n-k-1} \left[k - np \right] \\
&= \sum_{k=0}^{m} \binom{n}{k} k p^{k-1} (1-p)^{n-k-1} - n \sum_{k=0}^{m} \binom{n}{k} p^k (1-p)^{n-k-1} \\
&= \frac{n}{1-p} \sum_{k=0}^{m-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} - \frac{n}{1-p} \sum_{k=0}^{m} \binom{n}{k} p^k (1-p)^{n-k} \\
&= \frac{n}{1-p} \left[\mathbb{P}\left(\sum_{k=1}^{n-1} X_k \leq m-1 \right) - \mathbb{P}\left(\sum_{k=1}^{n} X_k \leq m \right) \right].
\end{aligned}$$

Consider the term in square brackets. The first probability is strictly smaller than the second, as the first event (at most $m-1$ successes in the first $n-1$ draws) implies the second one (at most m successes during all n draws), whereas the latter also includes the positive-probability event that $\sum_{k=1}^{n-1} X_k = m$. Hence, the derivative is negative, as we had to show.

Write a function $g : \{0, 1, \ldots, n\} \to \mathbb{R}$ as the sum of indicator functions:

$$\begin{aligned}
g &= g(n)\mathbf{1}_{\{0,\ldots,n\}} + (g(n-1) - g(n))\mathbf{1}_{\{0,\ldots,n-1\}} + \cdots + (g(0) - g(1))\mathbf{1}_{\{0\}} \\
&= g(n)\mathbf{1}_{\{0,\ldots,n\}} + \sum_{k=0}^{n-1} (g(k) - g(k+1))\mathbf{1}_{\{0,\ldots,k\}}.
\end{aligned}$$

Then

$$\mathbb{E}[g(X)] = g(n) + \sum_{k=0}^{n-1} (g(k) - g(k+1))\mathbb{P}(X \leq k).$$

If g is nonconstant, non-increasing, then $g(k) - g(k+1) \geq 0$ for all $k = 0, \ldots, n-1$, with at least one strict inequality. As shown above, the cumulative probabilities are strictly decreasing in the success probability p. So $\mathbb{E}[g(X)]$ becomes a strictly decreasing function of p: the higher the probability of success, the larger the probability that $g(X)$ achieves a low value. Of course, for non-decreasing functions the converse holds.

8.B Proof of Proposition 8.4.2

The only Nash equilibria not covered by (a), (b), and (c) are those with one player (player 1, say) choosing -1, one player (player 2, say) choosing $+1$, and the third player (player 3, say) mixing with probability $\lambda \in (0,1) \setminus \{\frac{1}{2}\}$.

Suppose, to the contrary, that such an equilibrium is the limit of a sequence of logit QRE $(p(\beta_n), q(\beta_n), r(\beta_n), \beta_n)_{n \in \mathbb{N}}$ where $\beta_n \to \infty$ and $(p(\beta_n), q(\beta_n), r(\beta_n), \beta_n)$ solves equations (8.13) to (8.15) for a logit QRE. In the selected equilibrium, both the (-1)-player and the $(+1)$-player choose their unique best response. By Lemma 3 in Turocy (2005, p. 251), $\beta_n(1 - p(\beta_n)) \to 0$ and $\beta_n q(\beta_n) \to 0$. Substituting this in the logit QRE condition (8.15) for the third player gives that

$$r(\beta_n) = \frac{1}{1 + \exp\left[-\beta_n(1 - p(\beta_n) - q(\beta_n))\right]} \to \frac{1}{2},$$

contradicting the assumption that $\lim_{n\to\infty} r(\beta_n) = \lambda \neq 1/2$.

It remains to show that the classes of equilibria in the proposition are indeed limits of a sequence of logit QREs.

(a): By symmetry, it suffices to show that the pure Nash equilibrium $(p, q, r) = (1, 1, 0)$ is the limit of a sequence of logit QREs.

Step 1: For each $\beta > 4$ there is a logit QRE (p, q, r, β) with $p = q \in (1/2, 1)$, and $r < 1/2$.

Proof of Step 1: Based on conditions (8.13) - (8.15) for a logit QRE and the substitution $p = q$, define for all $\beta > 0$ and $p \in [1/2, 1]$:

$$r(p,\beta) := \frac{1}{1+\exp[-\beta(1-2p)]},$$
$$f(p,\beta) := \frac{1}{1+\exp[-\beta(1-p-r(p,\beta))]}.$$

Let $\beta > 4$. We show that there is a solution $p^* \in (1/2, 1]$ to the equation $p = f(p,\beta)$. Substitution in (8.13) - (8.15) yields that $(p,q,r,\beta) = (p^*, p^*, r(p^*,\beta), \beta)$ is a logit QRE with the desired properties. Notice that

$$\begin{aligned}\frac{\partial f(p,\beta)}{\partial p} &= \frac{-\beta\exp\left[-\beta(1-p-r(p,\beta))\right]}{\left(1+\exp\left[-\beta(1-p-r(p,\beta))\right]\right)^2}\left(1+\frac{\partial r(p,\beta)}{\partial p}\right)\\ &= \frac{-\beta\exp\left[-\beta(1-p-r(p,\beta))\right]}{\left(1+\exp\left[-\beta(1-p-r(p,\beta_0))\right]\right)^2}\left(1+\frac{-2\beta\exp\left[-\beta(1-2p)\right]}{\left(1+\exp\left[-\beta(1-2p)\right]\right)^2}\right).\end{aligned}$$

Since $f(1/2,\beta) = 1/2$ and

$$\frac{\partial f(1/2,\beta)}{\partial p} = -\frac{\beta}{4}\left(\frac{2-\beta}{2}\right) > 1$$

for $\beta > 4$, it follows that $f(p,\beta) > p$ for p slightly larger than $1/2$. Moreover, $f(1,\beta) < 1$. Hence, by the Intermediate Value Theorem applied to $f(\cdot,\beta)$, $f(p^*,\beta) = p^*$ for some $p^* \in (1/2, 1)$.

Step 2: Let $\beta_0 > 4$ and let $p_0 \in (1/2, 1)$ solve $f(p_0, \beta_0) = p_0$. This is possible by Step 1. The function $f(p_0, \cdot)$ is strictly increasing on $[\beta_0, \infty)$.

Proof of Step 2: By definition of f, it suffices to show that the derivative of

$$\beta \mapsto \beta(1-p_0-r(p_0,\beta)), \qquad \beta \in [\beta_0,\infty)$$

is positive. This derivative equals

$$-\beta\frac{\partial r(p_0,\beta)}{\partial \beta} + 1 - p_0 - r(p_0,\beta). \tag{8.16}$$

Using $p_0 > 1/2$ and the definition of r, it follows that $\partial r(p_0,\beta)/\partial\beta < 0$, i.e., the function $r(p_0,\cdot)$ is strictly decreasing on $[\beta_0,\infty)$. Moreover, as $p_0 = f(p_0,\beta_0) > 1/2$, it follows from the definition of f that $1 - p_0 - r(p_0,\beta_0) > 0$. As $r(p_0,\cdot)$ is decreasing, this implies that $1 - p_0 - r(p_0,\beta) > 0$ for each $\beta \in [\beta_0,\infty)$. Therefore, the expression in (8.16) is positive.

Step 3: The pure Nash equilibrium $(p, q, r) = (1, 1, 0)$ is the limit of a sequence of QREs.

Proof of Step 3: Let $\beta_0 > 4$ and consider a QRE (p_0, q_0, r_0, β_0) as in Step 1. Set $\beta_1 = \beta_0 + 1$. By Step 2, $p_0 = f(p_0, \beta_0) < f(p_0, \beta_1)$. Moreover, $f(1, \beta_1) < 1$. By the Intermediate Value Theorem applied to the function $f(\cdot, \beta_1)$, there is a $p_1 \in (p_0, 1)$ with $p_1 = f(p_1, \beta_1)$. Conclude that there is a QRE (p_1, q_1, r_1, β_1) with

$$\begin{array}{lllll} p_1 & = & q_1 & = & f(p_1, \beta_1) > p_0, \\ r_1 & = & r(p_1, \beta_1), & & \\ \beta_1 & = & \beta_0 + 1. & & \end{array}$$

Repeating this construction allows us to define a sequence $(p_n, q_n, r_n, \beta_n)_{n \in \mathbb{N}}$ of solutions to (8.13) - (8.15) satisfying the conditions of Step 1 and with $\beta_n \to \infty$ and $(p_n)_{n \in \mathbb{N}}$ strictly increasing.

As $(p_n, q_n, r_n)_{n \in \mathbb{N}}$ is a sequence in the compact strategy space, we may assume without loss of generality that the sequence converges. Its limit (p, q, r) must be a Nash equilibrium (McKelvey and Palfrey, 1995). As $(p_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence in $(1/2, 1)$ and $(r_n)_{n \in \mathbb{N}}$ is a sequence in $(0, 1/2)$, it must be $p = q > 1/2$ and $r \leq 1/2$. The only Nash equilibrium of the game with these properties is $(p, q, r) = (1, 1, 0)$.

(b): By symmetry, it suffices to show that the Nash equilibrium $(p, q, r) = (1, 0, 1/2)$ is the limit of a sequence of logit QREs. The steps are similar to those in (a). Therefore, the proof is kept short.

Step 1: For each $\beta > 4$ there is a logit QRE (p, q, r, β) with $p \in (1/2, 1), q = 1 - p, r = 1/2$.

Proof of Step 1: Let $\beta > 4$. Based on the substitution $q = 1 - p$ and $r = 1/2$ in condition (8.13) for a logit QRE, define

$$g(p, \beta) := \frac{1}{1 + \exp\left[\beta(1/2 - p)\right]}.$$

We show that there is a solution $p^* \in (1/2, 1)$ to the equation $p = g(p, \beta)$. Substitution in (8.13) - (8.15) yields that $(p, q, r, \beta) = (p^*, 1 - p^*, 1/2, \beta), \beta)$ is a logit QRE with the desired properties. Notice that

$$\frac{\partial g(p, \beta)}{\partial p} = \frac{\beta \exp\left[\beta(1/2 - p)\right]}{\left(1 + \exp\left[\beta(1/2 - p)\right]\right)^2}.$$

Since $g(1/2, \beta) = 1/2$ and $\partial g(1/2, \beta)/\partial p = \beta/4 > 1$, it follows that $g(p, \beta) > p$ for p slightly larger than $1/2$. Moreover, $g(1, \beta) < 1$, so the Intermediate Value Theorem implies that $g(p^*, \beta) = p^*$ for some $p^* \in (1/2, 1)$.

Step 2: For each $p_0 \in (1/2, 1)$, the function $g(p_0, \cdot)$ is strictly increasing on $(0, \infty)$.

Proof of Step 2: Immediate from the definition of g.

Step 3: The Nash equilibrium $(p, q, r) = (1, 0, 1/2)$ is the limit of a sequence of logit QREs.

Proof of Step 3: Reasoning as in the proof of step 3 in part (a) allows us to construct a sequence $(p_n, q_n, r_n, \beta_n)_{n \in \mathbb{N}}$ of solutions to (8.13) - (8.15) satisfying the conditions of Step 1 and with $\beta_n \to \infty$ and $(p_n)_{n \in \mathbb{N}}$ strictly increasing. As $(p_n, q_n, r_n)_{n \in \mathbb{N}}$ is a sequence in the compact strategy space, we may assume without loss of generality that the sequence converges. Its limit (p, q, r) must be a Nash equilibrium (McKelvey and Palfrey, 1995). As $(p_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence in $(1/2, 1)$, $q_n = 1 - p_n$ and $r_n = 1/2$ for all $n \in \mathbb{N}$, it must be $p > 1/2, q = 1 - p, r = 1/2$. The only Nash equilibrium of the game with these properties is $(p, q, r) = (1, 0, 1/2)$.

(c): It follows by substitution that $(p, q, r, \beta) = (1/2, 1/2, 1/2, \beta)$ is a logit QRE for all $\beta \geq 0$. Consequently, the Nash equilibrium $(p, q, r) = (1/2, 1/2, 1/2)$ is the limit of a sequence of logit QREs with $\beta \to \infty$. $\square$

9 Coordinating to differentiate

Summary

In the previous chapter, we characterized the limiting behavior of a number of standard learning models for the minority game, a simple congestion game. In the current chapter, we give a critical account of an alternative learning model. We relate this model to standard learning models, and compare its predictions to experimental results.

9.1 Introduction

In the previous chapter, we have studied the limiting behavior of several well-known learning process for the minority game, a simple congestion game based on the El Farol bar problem of Arthur (1994). In the minority game, players have to choose between two alternatives, and players prefer to choose the alternative that is chosen by the smallest number of opponents. That is, players need to coordinate to differentiate. In the previous chapter, we have seen that the predictions of standard learning models differ markedly for the minority game, with some learning models predicting pure strategy equilibria, and others predicting equilibria with some mixing, making our theoretical understanding of play in such games incomplete.

Moreover, experimental results on such congestion games are sometimes hard to interpret. Though players are remarkably successful at learning to coordinate in congestion games, rapidly achieving a "magical" degree of coordination in the words of Kahneman (1988), regularities on the aggregate level generally conceal non-equilibrium behavior at the individual level. That is, even though aggregate play is close to the Nash equilibrium, individual players generally do not seem to play equilibrium strategies (see Ochs (1999) and references therein for results on market entry games, see e.g. Selten et al. (2007) for similar results on a related game). In addition, the effect of information on players' behavior remains unclear. In some cases, providing players with additional information leads to higher payoffs (e.g. Duffy and Hopkins, 2005), where standard learning models predict that this information should have no effect on play.

These experimental findings are hard to explain using standard learning models, such as the ones discussed in the previous chapter. The current chapter therefore discusses an alternative learning model, proposed in the literature on the minority game. In this model, players condition their behavior on a limited history of past outcomes, using so-called response modes that prescribe which action to take given the history of recent outcomes. Players decide on which response mode to use on the basis of its past performance. There are two important differences with standard learning models. Firstly, players do not take into account that their action affects the aggregate outcome. Secondly, it is assumed that players are endowed with a random selection of response modes.

These assumptions are natural in the current context. Congestion games, such as the route-choice games studied by Selten et al. (2007), often involve a large number of players, so that players may not account for the impact of their own action. Secondly, the assumption that players are endowed with a random selection of response modes is natural given that in the type of games studied here, there are no response modes that are a priori better than others: whether a response mode is successful (in the sense that it gives high payoffs) only depends on whether there are other players using an antagonistic response mode. For instance, Selten et al. (2007) identify two groups of players in their experiments on route-choice behavior. One group switches roads when it was crowded last period, while the other group then sticks with their choice, as they expect others to switch. The interaction between these two groups make that traffic is divided fairly evenly over the different roads, thus giving rise to the magical coordination Kahneman (1988) alluded to. It is thus the interaction between response modes that determine the success of response modes, so that it may well be realistic that players just use some simple heuristics or rules of thumb, rather than trying to identify the optimal strategy.

In the current chapter, we relate the learning model proposed in the minority game literature to the standard game-theoretic learning models, and compare its predictions to experimental results on congestion games. The contribution of this chapter is that it relates the literature on this learning model, which has been largely developed in physics, to the literature on learning in game theory and to the literature in experimental economics on congestion games. We have no intention of giving a comprehensive survey of the extensive minority game literature. For a collection of papers on the minority game, see `http://www.unifr.ch/econophysics/minority`, and see Challet et al. (2004) or Moro (2003) for an introduction to the field.

The outline of this chapter is as follows. We discuss the learning model proposed in the minority game literature in Section 9.2, and compare it to game-theoretic learning models. In Section 9.3, we discuss its main predictions. These predictions are compared to experimental results on congestion games in Section 9.4. Section 9.5 concludes.

9.2 Learning to play the minority game

9.2.1 The minority game

We briefly recall the definition of the minority game, and summarize the main results of the equilibrium characterization. For an elaborate discussion of the stage game and a full account of those results, see the previous chapter.

There is a set $N = \{1, \ldots, 2k+1\}$ of players, where $k \in \mathbb{N}$. Each player $i \in N$ has a set of pure strategies $A_i = \{-1, +1\}$. The payoffs only depend on the number of opponents choosing that action: for $b \in \{-1, +1\}$, there is a function

$$f_b : \{1, \ldots, 2k+1\} \to \mathbb{R}$$

that for each $\ell \in \{1, \ldots, 2k+1\}$ gives the payoff $f_b(\ell)$ to a player choosing action b when the total number of players choosing b equals ℓ. If action $b \in \{-1, +1\}$ is chosen by k players or fewer under a strategy profile $a \in \times_{j \in N} A_j$, we say that it is the *minority action* under a, otherwise it is the *majority action*.

The von Neumann-Morgenstern utility function of player $i \in N$ is then given by

$$u_i(a) = f_{a_i}\left(|\{j \in N \mid a_j = a_i\}|\right), \tag{9.1}$$

where $a = (a_j)_{j \in N} \in \times_{j \in N} A_j$. Payoffs are extended to mixed strategies in the usual way.

The function f_b, where $b \in \{-1, +1\}$, can have several forms. In the previous chapter, we only assumed that the function is (*i*) symmetric, i.e., $f_{-1} = f_{+1} = f$, (*ii*) monotonic, i.e., f is a strictly decreasing function. Here, to be in line with much of the minority game literature (e.g. Challet et al., 2004; Moro, 2003), we assume

$$\forall b \in \{-1, +1\} : \quad f_b(\ell) = \frac{2(k-\ell)+1}{2k+1}.$$

Clearly, this function satisfies the symmetry and monotonicity conditions above. To interpret this function, note that if the action profile is $a = (a_j)_{j \in N} \in \times_{j \in N} A_j$, with ℓ players choosing -1, and $2k + 1 - \ell$ players choosing $+1$, where $\ell \in \mathbb{N}$, the *aggregate action* $\sum_{i \in N} a_i$ is equal to $-\ell + (2k + 1 - \ell) = 2(k - \ell) + 1$. When $\ell \geq k + 1$, i.e., -1 is the majority action under the strategy profile a, this term is negative, and players who chose -1 ($+1$) receive a negative (positive) payoff.

In the previous chapter, we characterized the set of equilibria of the minority game. The game has a number of asymmetric pure Nash equilibria in which k players choose one option (the minority option) and $k + 1$ players choose the other one. In addition, there is a unique symmetric Nash equilibrium in which each player chooses each alternative with probability 1/2. Also, there are mixed Nash equilibria with k players choosing -1 with probability 1, k players choosing $+1$ with probability 1, and a player randomizing between -1 and $+1$ with any probability. Finally, there are a number of mixed Nash equilibria with more than one mixer.

The large number of equilibria—there is in fact a continuum of equilibria—makes it hard to predict which equilibrium will be played. As the equilibria in pure strategies cannot be Pareto-ranked or ordered in terms of risk-dominance, no particular pure Nash equilibrium can be singled out as being most salient (Schelling, 1960). Hence, without pre-play communication, players do not have enough information to implement a pure Nash equilibrium (cf. Menezes and Pitchford, 2006). While players could use common knowledge of rationality and symmetry to deduce and select the symmetric mixed Nash equilibrium (cf. Ochs, 1990; Meyer et al., 1992), this may raise an incentive problem, as players can earn a higher payoff than in the symmetric mixed Nash equilibrium if they manage to outsmart other players. Hence, players may try to find patterns in the play of others when the game is played repeatedly (cf. Arthur, 1994; Meyer et al., 1992). The learning model proposed in the minority game literature provides a way of formalizing this notion. We introduce this model in the next section.

9.2.2 Learning model

Time is discrete and indexed by $t \in \{0, 1, \ldots\}$. At each time t, the stage game is played. After each round of play t of the stage game, the players are informed of the aggregate action $A(t) := \sum_{i \in N} a_i(t)$, where $a_i(t) \in A_i = \{-1, +1\}$ is the action taken by player i at time t. We assume that players have a limited memory: they only retain the sequence of the minority actions in the previous m rounds,

h_m	$s_{i,1}$	$s_{i,2}$	$s_{i,3}$	$s_{i,4}$
$(-1,-1,-1)$	$+1$	-1	-1	$+1$
$(-1,-1,+1)$	-1	-1	$+1$	-1
$(-1,+1,-1)$	$+1$	-1	-1	$+1$
$(-1,+1,+1)$	-1	$+1$	-1	$+1$
$(+1,-1,-1)$	$+1$	$+1$	$+1$	$+1$
$(+1,-1,+1)$	-1	-1	$+1$	$+1$
$(+1,+1,-1)$	-1	-1	-1	$+1$
$(+1,+1,+1)$	-1	$+1$	-1	$+1$

Table 9.1. An example of a subset of response modes with $m = 3$ and $n_S = 4$ for some player $i \in N$. For instance, if the history of outcomes is $(-1,-1,-1)$, then response mode $s_{i,1}$ prescribes action $a_i = +1$, and response mode $s_{i,2}$ prescribes action $a_i = -1$.

where $m \in \mathbb{N}$. More specifically, in round t, players observe a *history* $h_m(t) = (-\text{sign}[A(\tau)])_{\tau\in\{t-m,t-m+1,\dots,t-1\}}$, where we note that $-\text{sign}[A(t)]$ indicates the minority action at time t: if there are fewer players choosing -1 ($+1$) at time t than there are players choosing $+1$ (-1), then $-\text{sign}[A(t)]$ is equal to -1 ($+1$).[1]

A *response mode* s assigns to each history $h_m \in \{-1,+1\}^m$ an action $a \in \{-1,+1\}$. That is, a response mode s prescribes which action $s(h_m(t)) \in \{-1,+1\}$ to take, for a given history of play $h_m(t)$ at time t. Note that a $s(h_m(t))$ does not depend on t, other than through $h_m(t)$: if $h_m(t) = h_m(t')$, then $s(h_m(t)) = s(h_m(t'))$. It is not hard to see that there are 2^{2^m} different response modes: there are 2^m possible signals h_m of length m, and for each signal, there are two possible actions. For memory length m, denote the set of all response modes by $\mathcal{S}^{(m)}$. An important assumption in the current learning model is that each player $i \in N$ is endowed with a subset S_i of the set of all possible response modes $\mathcal{S}^{(m)}$. All players are endowed with the same number of response modes: there exists $n_S \in \mathbb{N}$ such that $|S_i| = n_S$ for all $i \in N$. We assume that $n_S \geq 2$. For each player $i \in N$, the response modes in S_i are drawn uniformly at random from $\mathcal{S}^{(m)}$, independently across players. An example for $n_S = 4$ and $m = 3$ is given in Table 9.1.

When faced with a history, a player has to choose which of his response modes to use in the next period. Each player keeps a virtual score for each response mode in his endowment that reflects that response mode's past performance. The

[1] The history $h_m(t)$ at $t = 0$ is drawn uniformly at random from $\{-1,+1\}^m$.

virtual score of each response mode is updated after each time period, regardless of whether the response mode has been used or not. When a response mode would have correctly predicted the minority action, its virtual score is increased with the payoffs it would have earned, otherwise it is decreased with the same amount. More specifically, the *virtual score* of player $i \in N$ for response mode $s_i \in S_i$ at time $t > 0$ is given by:

$$p_{i,t}(s_i) = p_{i,t-1} - \left(\frac{s_i(h_m(t))}{2k+1}\right)[2(k - \ell(s_i(h_m(t)), t)) + 1]$$

where we recall that $s_i(h)$ is the action prescribed by response mode s_i when the history is h, and where $\ell(a, t)$ is the number of players choosing action a in round t. At $t = 0$, the virtual score of player i for s_i is $p_{i,t}(s_i) = 0$.

An important thing to note is that players do *not* take the effect of their action on the aggregate outcome into account. In determining the virtual score of a response mode, players only consider whether this response mode would have predicted the actual outcome correctly, neglecting the question whether playing this response mode would have affected the outcome. To see this, suppose that at time t, player i chooses $a_i = -1$, and that the total number of players choosing this action is $k + 1$, i.e., -1 is the majority action. Then, $2(k - (k + 1)) + 1 = -1$ would be added to all response modes prescribing action $a_i = -1$ (given the current history), and $-(2(k - (k + 1)) + 1) = +1$ would be added to all response modes prescribing $a_i = +1$. However, if player i would have chosen $a_i = +1$, the number of players choosing $a = +1$ would have been $k + 1$, and $+1$ would have been the majority action. This is an important assumption of the model, and we discuss its implications in Section 9.2.3.

The probability that a player chooses a response mode at a given time step is determined by its virtual score at that time, with the choice probabilities following the well-known logit choice rule. For $i \in N$, denote the response mode selected by player i at time t by $s_i(t)$. Then,

$$\forall s_i \in S_i : \quad \mathbb{P}(s_i(t) = s_i) = \frac{\exp\left[\beta\, p_{i,t}(s_i)\right]}{\sum_{s_j \in S_i} \exp\left[\beta\, p_{i,t}(s_j)\right]}, \tag{9.2}$$

where $\beta > 0$ is the logit parameter. The parameter β can be interpreted as the sensitivity of choice to marginal information. In the limiting case $\beta \to \infty$, players mix uniformly among the response modes with the highest virtual score. Otherwise, players choose response modes with lower virtual scores with positive probability, with a probability increasing in the virtual scores. Perhaps surprisingly, this additional noise may actually improve collective performance, as we discuss in Section 9.3.1.

9.2.3 Discussion

In this section, we discuss two of the most important assumptions of the learning model in the minority game model: (*i*) the assumption that all players are endowed with a random subset of response modes, (*ii*) the assumption that players update the virtual scores of response modes not used, without taking into account the effect of that response mode on the game's outcome. We discuss these assumptions in turn.

Response modes and heterogeneity

In the learning model proposed in the minority game literature, players base their action on the recent past, trying to discern patterns in their opponents' behavior, as in Arthur (1994). In the El Farol bar problem described by Arthur, players need to decide whether to go to a bar or not. Going to the bar is only pleasant if it is not too crowded. Arthur proposes that players condition their decision to go on attendance levels in the previous weeks: if the bar has been crowded for the last three weeks, say, they expect it to be crowded next week also. These mental models are mapped into actions: if a player expects the bar to be crowded, he will not go.

The response modes in the learning model of the minority game literature are a concise way of modeling this notion. An important question, however, is which response modes need to be included in the model. There are two possible approaches. Firstly, one could simply incorporate all possible response modes. However, if all possible response modes are included in the learning model, the strategy space becomes huge already for very simple games. Many different response modes are conceivable in a simple game such as the minority game, as illustrated by the list of examples in Arthur (1994).

A second possibility is to include only a selection of possible response modes. In that case, one could either make a selection based on behavioral assumptions, or let the subset of response modes be determined at random. In the first case, a natural choice is to include response modes that reflect beliefs about other players' actions, based on recent outcomes. The first approach is commonly taken in the economics literature (e.g. Erev and Rapoport, 1998; Selten et al., 2007), while the current learning model chooses the second approach. In the latter case, there are no restrictions on the types of response modes that players use.

This may seem to be a weak point of the model, as response modes need not have a sensible interpretation in the learning model of the minority game literature. However, in games such as the minority game, whether a response mode is reasonable *only* depends on the response modes used by others. Conversely, *any* response mode, whether it has a sensible interpretation or not, will work if opponents use response modes that recommend them to take the opposite action. For instance, in experiments on route-choice games, Selten et al. (2007) report that some subjects use a "direct" response mode, while other subjects use a "contrarian" response mode. Subject who use the former response mode will switch roads if they experienced congestion in the last period, while subjects using the contrarian response mode stick with their choice, as they expect other subjects to switch. The important point to note here is that the direct response mode is only sensible if there are players who use the contrarian response mode and vice versa. In such a case, agnosticism on the type of response modes that players use may well provide a more realistic model of players' reasoning processes than the more restrictive assumptions employed in different learning models. This offers an elegant solution to the dilemma signalled by Erev and Roth (1998, p. 873) that it is virtually impossible to include all possible behavioral rules, but that selection of specific rules bears the risk of "parameter fitting in a model with an enormous number of parameters". In the learning model proposed in the minority game literature, no response mode is ruled out on a priori grounds, while sensible behavioral rules evolve naturally, as the only criterion for a behavioral rule to be sensible in the minority game is that there are other players who follow a "contrarian" behavioral rule. Indeed, in Section 9.3.3, we show that under the current learning model, players will self-organize into groups that use different response modes in such a way that their actions cancel out to the extent possible.

However, this approach raises some questions. Firstly, one may ask why players are heterogeneous in their endowment of response modes. Perhaps more importantly, one could ask why players only consider a fixed number of response modes. Indeed, individual players have an incentive to increase the number of response modes they use, as that gives them an advantage over other players (Marsili et al., 2000). However, these assumptions are not uncommon in game-theoretic models of learning and bounded rationality. Possible justifications for such assumptions include that each player has different experiences prior to playing the minority game and therefore deems different response modes more reasonable than others (cf. Aumann, 1997; Fudenberg and Levine, 1998), and that boundedly rational players may prefer to just consider a subset of response modes that have worked well in the past, rather than considering all 2^{2^m} response modes (cf. Ellison and Fudenberg, 1993).

The law of simulated effect and boundedly rational players

Which response mode players choose from the set of response modes they are endowed with, is determined by the virtual score of each response mode. The learning process proposed in the minority game literature is closely related to the reinforcement learning model of Roth and Erev (1995) and Erev and Roth (1998). The main difference between the basic reinforcement learning model of Roth and Erev and the learning model of the minority game literature lies in the updating of the score of strategies or response modes not played. In the basic reinforcement learning model, the scores of these strategies are not updated, while in the learning model proposed in the minority game literature, the scores of all response modes are updated every period, as in hypothetical reinforcement learning or stochastic fictitious play (Fudenberg and Levine, 1998). The assumption that players also consider the payoffs to strategies or response modes not played seems to be reasonable. Camerer and Ho (1999) argue on the basis of theoretical arguments as well as on the basis of experimental findings that players obey not only the "law of *actual* effect", but also the "law of *simulated* effect", meaning that in reinforcement, not only payoffs from strategies that are actually used count, but also foregone payoffs from strategies not played.

However, for players to play according to the law of simulated effect, they need more information than for standard reinforcement learning. Under standard reinforcement learning, players only need to know the payoff to the action they choose. By contrast, to play according to stochastic fictitious play, players additionally need to know the payoff rule as well as the actions chosen by their opponents. Even in a game such as the minority game, where the players only need to know the aggregate choice of other players (and not their individual choices), calculating foregone payoffs of strategies not used may be too hard for players that are boundedly rational. In the learning model proposed in the minority game literature, players' bounded rationality is reconciled with the law of simulated effect by assuming that players do not take the effect of their own action on the global outcome into account. In that way, players can account for foregone payoffs of response modes not used, without having to do complicated calculations.

One may think that for a large number of players, it will not matter much whether players account for their own impact. However, due to the minority rule, there remains a systematic bias in the rewarding of response modes, even if the number of players is arbitrarily large. The reason is that the reward for a response mode that is currently played is systematically lower than that for the response modes that are not used. These latter response modes get a point if they prescribe the current minority side, even if they would have tipped the minority to the other side if they would have been played, so that they would have guessed wrong in reality. As the response mode that is actually played does not have this advantage, the response modes that are not played are systematically favored and hence results depend on whether players take the effect of their action on the aggregate outcome into account. This makes that players keep switching between response modes: over time, a response mode that is not played for some time will gather sufficiently many virtual points so as to be selected to be played, thus losing its advantage, until another response mode takes over again.

Interestingly, if players correct for this bias by allocating a small additional reward to response modes currently played, this does not happen, and in the long run, players use the same response mode in every period (in the limit $\beta \to \infty$). When m is sufficiently small, in each round, k players will choose one action, and $k + 1$ the other, as in the pure Nash equilibria of the game (Marsili et al., 2000).

The learning model proposed in the minority game literature thus combines features from several learning models in the literature on learning in games. However, it makes distinctly different predictions than game-theoretical learning models. To these predictions we now turn.

9.3 Predictions of the learning model

In this section, we discuss the main predictions on the learning model proposed in the minority game literature. Some results are obtained analytically, others by simulations. In the simulations, a given number of agents is endowed with a random subset of response modes, and results are obtained by averaging over different assignments. In the first two sections, we characterize the behavior of the model in terms of social efficiency and informational efficiency, and show that the two are intimately linked in this learning model. In Section 9.3.3, we discuss how the predictions of the model can be understood in terms of the formation of groups who use counteracting response modes.

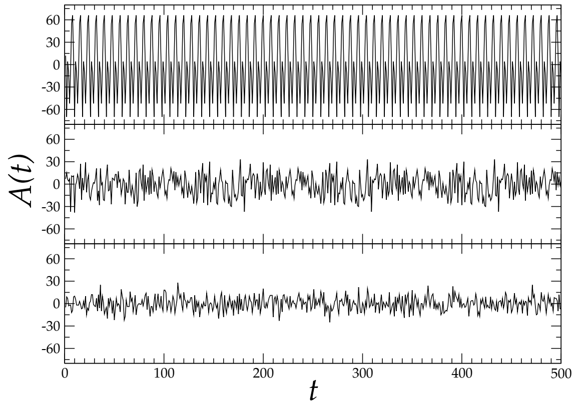


Figure 9.1. Time evolution of the aggregate action $A(t)$, with $2k + 1 = 301$ and $n_S = 2$. Panels correspond to $m = 2, 7, 15$ from top to bottom. Figure taken from Moro (2003).

9.3.1 Volatility

Simulations show that the aggregate action $A(t) := \sum_{i \in N} a_i(t)$ keeps fluctuating around 0, as can be seen in Figure 9.1. As the game is symmetric, the time average of $A(t)$ will be 0 in the long run (e.g. Challet and Zhang, 1997). More interesting is the behavior of the *volatility* $\sigma^2 := \langle A^2 \rangle$, where $\langle \cdot \rangle$ denotes the (time) average of a quantity. The volatility is a measure of the degree of efficiency (measured in terms of aggregate payoffs) achieved in a population. The higher the volatility, the larger the loss in aggregate payoffs: large fluctuations around 0 imply that the size of the minority is only small, as aggregate payoffs are proportional to $-\sum_i a_i(t)A(t) = -(A(t))^2$.

By simulations, it has been found that σ^2 is only a function of $\alpha := 2^m/(2k + 1)$ and n_S, where we recall that n_S is the number of response modes of each player (Savit et al., 1999). Figure 9.2 shows the volatility as a function of α. As can be seen in the figure, the volatility converges to the volatility exhibited in the symmetric mixed Nash equilibrium for $\alpha \to \infty$. With a large number of players (α small), overall performance is much worse. In fact, the volatility is of order $(2k + 1)^2$, so that the size of the group of players choosing the minority action is much smaller than k. At intermediate values of α, volatility is low, and it attains a minimum at $\alpha_c(n_S) \cong n_S/2 - 0.66$ (Marsili et al., 2000). Hence, at intermediate values of α, players are able to coordinate their actions and perform better collectively than under

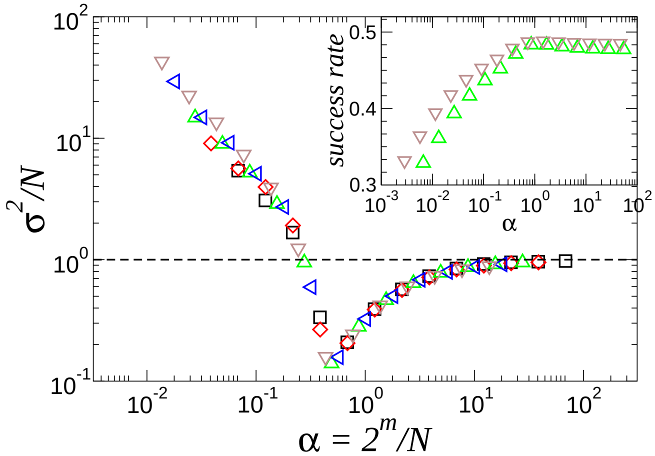


Figure 9.2. Volatility as a function of the order parameter α for $n_S = 2$ and different number of players $N := 2k + 1 = 101, 201, 301, 501, 701$ (□, ◊, △, ◁, ▽, respectively). The critical value α_c is the value of α for which the volatility is at a minimum. Inset: players' average success rate as function of α (not discussed here, see Moro, 2003). Figure taken from Moro (2003).

the symmetric mixed Nash equilibrium. This means that players can somehow exploit the available information to reduce σ^2 relative to the symmetric mixed Nash equilibrium. Note that this is not the result of some form of cooperative behavior of the players: agents are selfishly maximizing their own payoffs, and for intermediate values of α, this leads to higher aggregate payoffs. However, coordination is not complete under the current learning model. The aggregate payoff is maximized if players play according to one of the pure Nash equilibria of the game, with k players choosing the minority action. In that case, almost half of the players are in the minority, and $\sigma^2/(2k + 1) = 1/(2k + 1)$. Players come close to this optimum at $\alpha = \alpha_c$, but they never reach it.

Strikingly, global efficiency is enhanced for certain values of α when players do not always choose the response mode with the highest number of virtual points, i.e., when $\beta < \infty$ in Equation (9.2). It can be shown that for $\alpha < \alpha_c$, when the volatility is much higher than under the benchmark of the symmetric Nash equilibrium, volatility *decreases* when the noise level $1/\beta$ *increases*. For $\alpha > \alpha_c$, the value of β does not affect the level of volatility (Cavagna et al., 1999). The explanation is that under the current learning model, players form herds when $\alpha < \alpha_c$. The parameter α is a measure of the total number of response modes relative to the number of players. When $\alpha < \alpha_c$, there are few response modes relative to the number of players. In that case, players have to herd at a limited number of response modes, leading to a large number of players choosing the same alternative (see Section 9.3.3). Decreasing β is then equivalent to slowing down the updating of virtual scores for response modes. A finite β therefore acts as a brake against overreaction.[2]

9.3.2 Information and efficiency

As discussed in the previous section, players seem to be able to coordinate reasonably well for some parameter configurations. The only way players can interact is through the history of play. This observation led some authors to study the information contained in the history of play. The information content of the history of play, or the degree of predictability can be measured by (Challet and Marsili, 1999):

$$H := \frac{1}{2^m} \sum_{\nu=1}^{2^m} \langle A(t+1) \mid h_m(t) = \nu \rangle^2,$$

where $\langle A(t+1)|h_m(t) = \nu\rangle$ is the time average of the aggregate action conditional on a given history of play. Loosely speaking, H measures the information in the time series of $A(t)$. If $A(t+1)$ and $h_m(t)$ are independent, then $H = 0$. If $H > 0$, then the signal $A(t)$ contains information. It can be shown that players under the current learning model minimize the degree of predictability (Marsili et al., 2000). Depending on the value of α, players are more or less successful in doing that. At α_c, the system changes from an informationally efficient phase with low aggregate payoffs ($H = 0$, σ^2 large) to an information-rich phase with high aggregate payoffs ($H > 0$, σ^2 small). In the informationally efficient phase, aggregate payoffs are lower than under the symmetric mixed Nash-equilibrium. By contrast, in the information rich phase, players manage to coordinate and aggregate payoffs are higher than under the symmetric mixed Nash equilibrium.

This transition between the informationally efficient and the information rich phase, or equivalently between the phase with low aggregate payoffs and the phase with high aggregate payoffs, is central to the current learning model. At this transition, there is a qualitative change in collective behavior, while the principles

2 This result is reminiscent of the findings of Goeree et al. (2007) who show that in a social learning model, payoff-dependent noise in the decision process can break the cascades that would otherwise result.

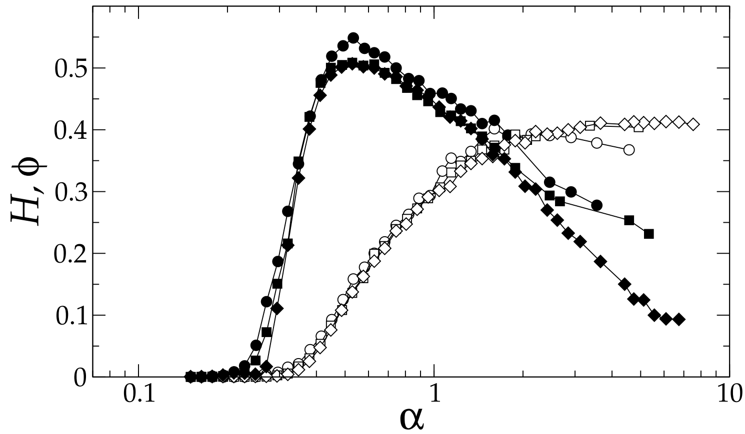


Figure 9.3. Information H (open symbols) and fraction of frozen players ϕ (full symbols; not discussed here, see Moro, 2003) as a function of the control parameter $\alpha = 2^m/(2k+1)$ for $n_S = 2$ and $m = 5, 6, 7$ (circles, squares and diamonds, respectively). Figure taken from Moro (2003).

behind the behavior of individual players remain unchanged. For all values of α, players try to outsmart each other, but for low values of α, they are on average less successful. In the next section, we discuss the interpretation of α.

9.3.3 Response modes and their antagonists

We have seen that the qualitative behavior of the system depends mainly on $\alpha = 2^m/(2k+1)$. Moreover, for some values of α, players are much more successful in coordinating behavior than for other values. What is the feature of the model underlying this behavior? We address this question in the current section. The answer to this question points to an intuitive interpretation of the model's results in terms of groups using counteracting response modes.

The minority rule forces players to differentiate: if all players choose the same response mode, all players obtain negative payoffs. As there are 2^{2^m} possible response modes for $2k+1$ players, one would expect that players succeed in differentiating if $2k+1$ is much smaller than 2^{2^m}, and be unsuccessful when $2k+1$ exceeds 2^{2^m}. Hence, one would expect a qualitative change when $2k+1$ is of order 2^{2^m}, rather than of order 2^m, as observed. To explain why the transition occurs when $2k + 1$ is of order 2^m, we need some more definitions. For each $s, s' \in \mathcal{S}^{(m)}$, define

$$H^{(m)}(s, s') := \sum_{h \in \{-1,+1\}^m} |s(h) - s'(h)|$$

to be the Hamming distance between s and s'. Then, let

$$D^{(m)}(s, s') := \frac{1}{2^m} H^{(m)}(s, s')$$

be the *normalized distance* between s and s'. If $D^{(m)}(s, s') = 1$, i.e., s and s' prescribe different actions for each possible history of play, we say that s and s' are *anti-correlated*. When $D^{(m)}(s, s') = 1/2$, we say that they are *uncorrelated*. The reason that the transition occurs when $2k+1$ is of order 2^m is that two response modes only give rise to distinctively different behavior if they are anti-correlated or uncorrelated. It can be shown analytically that for every response mode, the number of response modes that are anti-correlated or uncorrelated with that response mode is $2 \cdot 2^m / n_S$ (Challet and Zhang, 1998). Hence, the qualitative behavior depends on 2^m, not 2^{2^m}.

This leads us to an intuitive interpretation of the model's results in terms of the interplay between groups using different response modes. Let s be a response mode, and let $\bar{s}$ be the response mode that is anti-correlated with s. Suppose N_s players use the response mode s in a given time period, while $N_{\bar{s}}$ players use the anti-correlated response mode $\bar{s}$ in that period. If N_s is approximately equal to $N_{\bar{s}}$ for all anti-correlated pairs $(s, \bar{s})$ of response modes, then the actions of players using these response modes effectively cancel and the volatility will be small.

Hence, it would be optimal if the group of players that use a certain response mode is of about the same size as the group that uses the "antagonistic" response mode. However, this is not always possible, as the dimension of the space of response modes is fixed by the parameter m. Hence, when the number of players is large, the number of response modes they use will be larger than 2^m, so that players are forced to use response modes that are positively correlated. This gives rise to herding effects and large volatility when α is small. For somewhat larger values of m (for a fixed number of players), players use response modes that are either uncorrelated or mutually anti-correlated. In that case, players spread more or less evenly over both actions in each period. Finally, when m is very large relative to the number of players, the number of players using a given response mode will only be small, so that players act more or less independently (Moro, 2003). However, aggregate payoffs are still higher than under the benchmark of the symmetric mixed Nash equilibrium, as there always exist pairs of players that follow anti-correlated response modes (Challet and Zhang, 1998).

9.4 Comparison to experimental results

In this section, we discuss some experiments on the minority game and related congestion games. In addition to the minority game, we focus on market entry games and route-choice games. First, we briefly introduce the latter two classes of games. We then present some experimental results, and discuss whether the learning model proposed in the minority game literature could explain these results.

The market entry game (Selten and Güth, 1982) has been studied extensively in economics (see Ochs (1999) and references therein; see Duffy and Hopkins (2005) for a recent contribution). In a market entry game, N players must decide independently and simultaneously to enter a market with a fixed capacity $c < N$ or to stay out. Players who enter the market receive a payoff that decreases in the number of entrants. The payoff of players who stay out of the market is commonly taken to be constant. The game generally has a large number of Nash equilibria, both in pure and in mixed strategies. Depending on the exact form of the payoff function, there may even be a continuum of equilibria. Pure Nash equilibria may be payoff-symmetric or payoff-asymmetric, and strict or non-strict, depending on the choice of parameters. For the payoff functions commonly studied, the (expected) number of entrants is between $c - 1$ and c in equilibrium. An important difference between the market entry game and the minority game is that in the latter game, the payoffs of both actions are subject to congestion, while in the market entry game, players can choose between an action with constant payoffs (staying out) and an action whose payoffs are subject to congestion.

Route-choice games are closer to the minority game in that the payoffs of all actions are subject to congestion. In a route-choice game, players choose between two or more roads. The payoffs of choosing one of these roads decrease in the number of other players who choose that road. In equilibrium, players divide themselves over the roads in such a way that travelling times and hence payoffs are equalized. These games have been studied experimentally by a number of authors (see Selten et al., 2007, and references therein). An important difference with the minority game is that the pure Nash equilibria of the route-choice game are payoff-symmetric, and that they are strict.

We now turn to some experimental work on market entry games, route-choice games and the minority game, and discuss whether experimental findings can be explained by the learning model proposed in the minority game literature. We focus on two issues: aggregate behavior versus individual play and the effect of information on players' behavior. We discuss these issues in turn.

Aggregate behavior versus individual play

A robust finding in experiments on games in these classes is that subjects quickly achieve a "magical" degree of coordination (in the sense that aggregate payoffs are high). However, individual players generally do not play equilibrium strategies. For instance, while Erev and Rapoport (1998) find that the number of entrants in a market entry game rapidly converges to the equilibrium value, they also observe large between- and within-subject variability, which does not diminish with experience. This is a common finding in experiments on market entry games (Ochs, 1999, p. 169).[3] Similarly, in their experiments on route-choice games, Selten et al. (2007) observe that the mean number of drivers on the different roads is very close to the equilibrium number, while large fluctuations in individual behavior persist until the end of the experiment. Similar experimental results have been reported for the minority game (Bottazzi and Devetag, 2007; Chmura and Pitz, 2006). In all cases, the hypothesis that fluctuations can be explained by a symmetric mixed Nash strategy equilibrium of the game can be rejected. These results cannot easily be explained with standard learning models, as these models typically predict convergence to the pure Nash equilibria of such games or to Nash equilibria with at most one mixer (see Duffy and Hopkins (2005) and the previous chapter for a discussion of the predictions of different learning models for the market entry game and the minority game, respectively).

Some authors attempt to reconcile aggregate "equilibrium" behavior in experiments with individual non-equilibrium play by conjecturing that subjects may use counteracting behavioral rules (Bottazzi and Devetag, 2007; Chmura and Pitz, 2006; Erev and Rapoport, 1998; Rapoport et al., 2000; Selten et al., 2007; Zwick and Rapoport, 2002). For instance, Bottazzi and Devetag (2007) find that there is considerable heterogeneity in players' behavior in their experiments on the minority game. They show that it is not the heterogeneity per se which determines the players' success in coordinating, rather, it is the interaction between these different behavioral rules that players can successfully coordinate on choosing different actions. These findings are in line with the predictions of the learning model of the minority game literature that players self-organize in groups that use coun-

[3] An exception is Duffy and Hopkins (2005) who find that subjects coordinate on one of the pure Nash equilibria of the market entry game after a large number of rounds when they are given feedback on others' choices.

teracting response modes, thus reconciling aggregate equilibrium behavior and individual non-equilibrium play.

In most experiments, it is not fully clear which behavioral rules subjects employ. For example, Selten et al. (2007) are unable to classify 42% of the subjects in terms of the behavioral rules they use in their route-choice experiments, while Zwick and Rapoport (2002) cannot classify the behavior of some 60% of their subjects in their experiments on the market entry game. This leaves open the possibility that subjects use some response modes that may not have an intuitive interpretation (and are thus not recognized by the experimenters) but that nevertheless perform well, since there are players using counteracting response modes, as predicted by the current learning model (see Section 9.2.3 and 9.3.3). A systematic study of the different response modes used by experimental subjects seems needed. Indeed, Zwick and Rapoport (2002) conclude that there is a need "to reorient research on interactive decision making to individual differences, identify patterns of behavior shared by subsets of players . . . , and then attempt to account for aggregate behavior in terms of the behavior of the clusters of players that form these aggregates".

Effect of information

The effect of information on players' behavior in such games remains a puzzle. Two dimensions of information have been investigated in the experimental literature.

A first dimension that has been studied is how behavior depends on the information subjects have on others' choices. Players can be provided with information only on the payoff rule and aggregate behavior in the past rounds or may be informed additionally of the individual choices of all other players. Although for a wide range of learning models including the reinforcement learning model and the learning model studied in the minority game literature, this should not affect results, in many experimental studies, behavior differs qualitatively depending on the information players have. For instance, in experiments on market entry games, Duffy and Hopkins (2005) find that behavior becomes less random when players are informed of other players' choices. In market entry games, this could be explained by the fact that providing players with more information allows them to use repeated game strategies, as the additional information allows players to signal their commitment to entering the market. While for the market entry game, such a signalling strategy pays off, this is not the case for the minority game and route-choice games. For instance, suppose that k players in the minority game commit to action $a = -1$, and k players commit to action $a = +1$. The remaining player will not be deterred from choosing either of those actions by the commitment of other players, nor does the commitment of these players guarantee them a positive payoff.[4] Nevertheless, also in the minority game and route-choice games, players switch less often between different actions when they are provided with information on the choices of others (Bottazzi and Devetag, 2007; Selten et al., 2007). This could be explained under the learning model of the minority game literature if the additional information induces players to account somehow for their impact on the aggregate action (see Section 9.2.3), but it is not clear why this would be the case.

A second dimension of information that has been studied in the literature refers to the salience of information on the recent history of play. Bottazzi and Devetag (2007) provide players with a string of past outcomes of varying length. When players are provided with information on play in more rounds than just the previous one, aggregate payoffs are significantly higher. Bottazzi and Devetag find that a longer history allows players to correlate their behavior over a longer time period. Notably, aggregate payoffs are highest in a treatment where players are provided with information on several rounds, and play is characterized by a substantial lack of short-range correlations between own current and past actions. Hence, players seem to exploit the additional information to improve their payoffs.

All together, these experimental studies lend some support to the learning model proposed in the minority game literature. However, the question how information influences play in congestion games has still not been satisfactorily answered. It would be interesting to compare players' behavior under different informational treatments in different congestion games. While most learning models make similar predictions for the different congestion games discussed here, intuitively, one would expect that information will play a different role in these games, as emotions like envy and regret will be more important in some games than in others, and also the scope for repeated-game strategies differs across games. Such a systematic comparison would allow one to better separate the learning effects from possible repeated-game and behavioral effects.

4 A repeated-game strategy that is effective in the minority game is one in which players "take turns": players alternately choose each of the two actions in such a way that each player is in the minority roughly half of the time. Indeed, Helbing et al. (2005) find some evidence of such behavior in their experiments on route-choice games with small groups, but it is unlikely that players will be able to successfully play according to such a repeated-game equilibrium when the number of players is large.

9.5 Conclusions

In this chapter, we have given a critical account of the learning model proposed in the minority game literature, and related it to standard learning and evolutionary models in economics, showing that it shares quite a few features with these models. Still, the predictions of this learning model are markedly different from the predictions from other models. We have argued that these predictions are in line with a number of experimental results on the minority game and related games which cannot be explained by other learning models.

However, our understanding of learning in such games is still complete. For instance, the effect of information on play is unclear. An interesting direction for further research would be to systematically vary players' information in experiments on different congestion games such as the minority game and the market entry game, and to compare play under the different information treatments and across games. While most learning models provide similar predictions for these games, intuitively, one would expect that information may have different effects in these games, as in some games, repeated-game strategies or emotions may play a larger role than in others. Such an experiment may help shed light on the question which learning model is appropriate in such games.

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Samenvatting

Netwerken en leerprocessen binnen de speltheorie

Dit proefschrift handelt over twee onderwerpen, netwerken en speltheorie, en leerprocessen in spelen. Deel I van dit proefschrift kijkt naar netwerken. Netwerken spelen een belangrijke rol binnen de economie. Ten eerste geven netwerken toegang tot zaken als informatie en kapitaal. Empirische studies laten bijvoorbeeld zien dat veel mensen een baan vinden via hun vrienden. Ten tweede wordt het gedrag en welzijn van een individu vaak meer beïnvloed door degenen met wie hij een directe relatie heeft dan door het gedrag van de gehele bevolking. Wanneer iemand bijvoorbeeld een communicatietechnologie kiest, kijkt hij vooral welke communicatietechnologie zijn collega's, familieleden of vrienden gebruiken.

In dit proefschrift nemen we aan dat economische agenten zoals individuen en bedrijven gepositioneerd zijn op een netwerk. Elke agent, ook wel speler genoemd, speelt een gegeven spel met zijn buren in het netwerk. In het voorbeeld hierboven over de keuze van een communicatietechnologie speelt ieder individu een coördinatiespel met zijn buren. In dit geval zijn de buren van een speler de mensen met wie hij wil kunnen communiceren, bijvoorbeeld zijn collega's. Iedere speler moet een keuze maken voor een communicatietechnologie. Wanneer hij dezelfde technologie kiest als een buurspeler, krijgt hij een positieve uitbetaling die hetzelfde is voor alle technologieën, anders ontvangt hij een uitbetaling gelijk aan nul. Zijn totale uitbetaling is gelijk aan de som van de uitbetalingen over de verschillende interacties met zijn buren. Een speler zal dan die technologie kiezen die door het grootste aantal buren gekozen wordt; zie Figuur 1. Echter, welke technologie zijn buren kiezen, hangt weer af van de technologie die hun buren

		j	
		X	Y
i	X	1,1	0,0
	Y	0,0	1,1

k_i	Totale uitbetaling
0	0
1	1
2	2
...	...
n_i	n_i

Figuur 1. Spelers kunnen kiezen tussen twee technologieën, X en Y. De uitbetalingsmatrix links geeft de uitbetalingen van het coördinatiespel tussen speler i en één van zijn buren, j. De tabel rechts geeft de totale uitbetaling van speler i wanneer hij dezelfde technologie kiest als k_i van zijn n_i buren, waar $k_i = 0, 1, \ldots, n_i$.

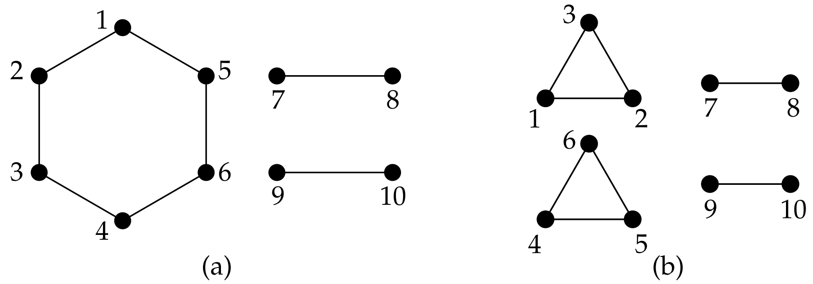
kiezen, enzovoort. Om te bepalen welke technologie hij moet kiezen, moet een speler dus ook de mogelijke keuzes van de buren van zijn buren en de buren van de buren van zijn buren etcetera beschouwen. Een complicatie hierbij is dat, hoewel een speler vaak wel zijn directe omgeving in het netwerk kent, hij zelden informatie heeft over de structuur van het gehele netwerk. Een speler zal dus verwachtingen moeten vormen over de netwerkstructuur en de interacties tussen de verschillende spelers.

Bovenstaand voorbeeld, hoe gestileerd ook, geeft een goed inzicht in de factoren die van belang zijn als we kijken naar de interacties van economische agenten in een netwerkcontext. Ten eerste handelen economische agenten vaak *strategisch*, dat wil zeggen, ze kiezen de actie die optimaal is gegeven het gedrag van andere agenten. In het voorbeeld hierboven kiezen agenten de communicatietechnologie die hen de hoogste uitbetaling geeft, gegeven de keuzes van hun buren. Ten tweede zijn de economisch relevante netwerken over het algemeen groot en hebben ze een complexe structuur. Bovendien veranderen ze voortdurend van structuur. Zo zijn samenwerkingsverbanden in onderzoek en ontwikkelingen in sectoren als de biotechnologie-sector erg flexibel om te kunnen reageren op nieuwe (technologische) ontwikkelingen. Om deze redenen zullen agenten over het algemeen *onvolledige informatie* hebben over het netwerk waartoe ze behoren. In Deel I van dit proefschrift kijken we daarom naar de interactie tussen onvolledige informatie en strategische interactie in een netwerkcontext.

In Hoofdstuk 3 en 4 kijken we naar de gevoeligheid van speltheoretische voorspellingen voor aannames over de verwachtingen van spelers over hun netwerk. In het voorbeeld over de keuze van communicatiebeslissingen hierboven is het belangrijk wat voor verwachtingen spelers hebben over het gedrag van hun buren, de buren van hun buren, enzovoort. Meer algemeen, wanneer spelers onvolledige informatie hebben over de netwerkstructuur moeten ook de verwachtingen en de informatie van spelers over hun netwerk gemodelleerd worden. We nemen aan dat het netwerk waarop spelers gepositioneerd zijn getrokken wordt uit een klasse netwerken volgens een bepaalde kansmaat. Deze kansmaat vormt de (gezamenlijke) prior van spelers. Daarnaast hebben spelers lokale informatie over hun netwerk: ze kennen hun graad in het netwerk, dat wil zeggen, het aantal buren dat ze hebben in het netwerk. Dit is private informatie, d.i., het type van een speler is zijn graad in het netwerk.

Hoofdstuk 3 handelt over Bayesiaanse netwerkspelen. Bayesiaanse netwerkspelen zijn Bayesiaanse spelen waarin spelers gepositioneerd zijn op een netwerk en onvolledige informatie hebben over het netwerk. De vraag die we in dit hoofdstuk proberen te beantwoorden is welke aspecten van een prior van belang zijn voor speltheoretische voorspellingen. Meer specifiek, neem twee priors. We onderzoeken onder welke condities op deze priors het zo is dat voor elke uitbetalingsfunctie, voor elk Bayesiaans-Nash-evenwicht in een spel met deze uitbetalingsfunctie waarin de spelers één van deze priors hebben er een bijna-evenwicht bestaat in het corresponderende spel waarin de spelers de andere prior hebben zodanig dat de verwachte uitbetalingen ongeveer hetzelfde zijn (en vice versa). In dat geval zeggen we dat de twee priors vergelijkbare strategische voorspellingen doen: de verwachte uitbetalingen onder de (bijna-)evenwichten onder de twee priors liggen dicht bij elkaar voor elke mogelijke uitbetalingsfunctie. We laten zien dat een noodzakelijke en voldoende voorwaarde voor twee priors om vergelijkbare strategische voorspellingen te doen is dat ze weinig verschillen in termen van de kansverdeling over lokale gebeurtenissen, dat wil zeggen, gebeurtenissen die betrekking hebben op de typen van een speler en zijn buren.

Dit resultaat heeft twee belangrijke implicaties. Ten eerste betekent dit resultaat dat het voldoende is om de distributie van spelerstypen en de correlatie tussen de typen van buren te variëren om alle mogelijke strategische uitkomsten te verkennen. Aan de ene kant betekent dit dat het niet voldoende is om de distributie van spelerstypen te variëren, zoals tot nu toe vaak gebeurt in de literatuur. Aan de andere kant beperkt het ook de verzameling priors die men hoeft te bekijken: priors hoeven slechts langs twee dimensies gevariëerd te worden, namelijk in termen van de typedistributie en de correlaties tussen de typen van buren. De tweede implicatie van dit resultaat is dat Bayesiaanse netwerkspelen gezien kunnen worden als een verzameling van overlappende lokale spelen, zodat we ons niet hoeven bezig te houden met de niet-lokale aspecten van de verwachtingen van spelers.



Figuur 2. De netwerken in (a) en (b) zijn identiek in termen van hun lokale eigenschappen. In beide netwerken zijn er 6 knopen (spelers) met graad 2 die alleen verbonden zijn met andere knopen met graad 2, en 4 knopen met graad 1 die alleen verbonden zijn met andere knopen met graad 1.

Meer concreet, beschouw de netwerken in in Figuur 2(a) en (b), en beschouw de volgende twee priors. De eerste prior geeft kans 1 aan de netwerken die isomorf[1] zijn aan het netwerk in Figuur 2(a), waarbij elk netwerk in de isomorfieklasse gelijke kans heeft. De andere prior geeft kans 1 aan de netwerken die isomorf zijn aan de netwerken in Figuur 2(b), waarbij weer elk netwerk in de betreffende isomorfieklasse dezelfde kans heeft. Deze priors zijn duidelijk heel verschillend in de kansen die ze toewijzen aan niet-lokale gebeurtenissen—bijvoorbeeld aan individuele netwerken—maar zijn identiek in de kansen die ze toewijzen aan lokale gebeurtenissen, zoals de kans dat een speler met een gegeven graad een buur heeft met graad 2. Onze resultaten laten zien dat deze priors identiek zijn in termen van hun speltheoretische voorspellingen, ook al zijn de netwerken die een positieve kans hebben onder beide priors erg verschillend.

Hoofdstuk 4 behandelt netwerkspelen waarin er onzekerheid is over het aantal spelers in het netwerk, naast onzekerheid over de structuur van het netwerk. Het is redelijk om aan te nemen dat er onzekerheid bestaat over het aantal spelers in een netwerk: wanneer spelers onvolledige informatie hebben over de netwerkstructuur zullen ze vaak ook niet precies weten hoeveel spelers tot het netwerk behoren. In dit geval is de spelersverzameling niet algemeen bekend, waardoor deze klasse spelen geen onderklasse is van de verzameling Bayesiaanse spelen, in tegenstelling tot de klasse netwerkspelen die in Hoofdstuk 3 bestudeerd worden. We nemen nu aan dat het netwerk getrokken wordt uit de klasse van alle

1 Twee netwerken zijn isomorf wanneer ze gedefinieerd zijn op dezelfde verzameling spelers en wanneer de spelers op dezelfde manier verbonden zijn.

netwerken met een eindig aantal spelers. Wanneer er onzekerheid is over het aantal spelers in het netwerk, is de voorwaarde die we afgeleid hebben in Hoofdstuk 3 voor twee priors om vergelijkbare voorspellingen te geven niet langer voldoende. We laten zien dat twee priors vergelijkbare voorspellingen geven in deze klasse spelen dan en slechts dan als (*i*) de priors vergelijkbare kans toekennen aan lokale gebeurtenissen, (*ii*) een speler met grote kans een type heeft zodanig dat zijn conditionele verwachtingen over de typen van zijn buren bijna hetzelfde zijn onder de twee priors, en dat zijn buren met hoge kans (gegeven het type van de speler) een type hebben zodanig dat hun conditionele verwachtingen over de typen van hun buren bijna hetzelfde zijn onder de twee priors, en dat met hoge kans (gegeven hun type) de conditionele verwachtingen van hun buren bijna hetzelfde zijn, enzovoort.

Conditie (*i*) komt overeen met de conditie die we afgeleid hebben in Hoofdstuk 3. Conditie (*ii*) is nieuw. De reden dat we deze conditie nodig hebben wanneer we onzekerheid over het aantal spelers in het netwerk toelaten is dat de hogere-orde verwachtingen van spelers nu een rol kunnen spelen in de zin dat priors nu ook gevoelig kunnen zijn voor gebeurtenissen die een lage (ex ante) kans hebben. Dat is, gebeurtenissen die een lage kans hebben kunnen een groot effect hebben op speltheoretische uitkomsten via de conditionele verwachtingen van spelers. Beschouw een verzameling typen waarvoor de conditionele verwachtingen sterk verschillen onder twee priors. Een speler die één van die typen heeft, volgt dan mogelijk een andere strategie onder de twee priors. In dat geval kunnen deze typen andere typen "infecteren" via de conditionele verwachtingen: een speler kan het waarschijnlijk achten (gegeven zijn type) dat zijn buren het waarschijnlijk achten (gegeven hun type) dat ... de conditionele verwachtingen van hun buren heel verschillend zijn onder de twee priors. Zelfs wanneer de verzameling van typen waarvoor de conditionele verwachtingen onder beide priors sterk verschillen een lage kans heeft (ex ante), kan de verzameling geïnfecteerde typen een hoge kans hebben, zodat de evenwichten onder beide priors sterk zullen verschillen in termen van verwachte uitbetalingen. Dit geval wordt uitgesloten door conditie (*ii*).

Priors kunnen alleen gevoelig zijn voor gebeurtenissen die een lage (ex ante) kans hebben wanneer er onzekerheid is over het aantal spelers in het netwerk. Wanneer er onzekerheid is over het aantal spelers kan de verzameling typen die positieve kans hebben aftelbaar oneindig zijn, terwijl de typeverzameling eindig is wanneer het aantal spelers algemeen bekend is.[2] We laten zien dat, wanneer het

[2] Het type van een speler is zijn graad in het netwerk. Wanneer het aantal spelers n is, dan is de kans 0 dat de graad van een speler in het netwerk groter is dan $n-1$, zodat de verzameling typen met positieve kans eindig is.

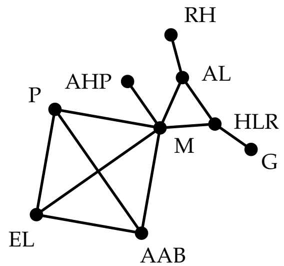
aantal spelers algemeen bekend is, conditie (*i*) conditie (*ii*) impliceert, maar dat in andere gevallen conditie (*i*) en conditie (*ii*) beide noodzakelijk zijn.

Het resultaat in Hoofdstuk 4 laat zien dat hogere-orde verwachtingen een belangrijke rol kunnen spelen in netwerkspelen. Bij de afleiding van de resultaten wordt gebruik gemaakt van een afbeelding van een stochastisch netwerk van *spelers* op een vaste interactiestructuur van *typen*. Vervolgens gebruiken we de equivalentie die in eerder werk is vastgesteld tussen spelen met lokale interacties en spelen met onvolledige informatie om gebruik te kunnen maken van concepten uit de literatuur van spelen met onvolledige informatie en hogere-orde verwachtingen.

Gemotiveerd door het belang van correlatie tussen de graden van buurspelers in netwerkspelen ontwikkelt Hoofdstuk 5 een model voor de verwachtingen van spelers waarin correlatie mogelijk is tussen de graden van buren. De meeste modellen die tot dusverre in de speltheoretische literatuur gebruikt worden nemen aan dat de graden van spelers (bijna) onafhankelijk zijn. Zoals aangetoond in Hoofdstuk 3 en 4 kunnen de evenwichten die onder priors die voldoen aan die aanname gevonden worden sterk verschillen van de evenwichten onder priors met correlatie tussen spelerstypen, zelfs wanneer de distributie van typen gelijk is onder beide klassen van priors. Bovendien zullen in veel economisch relevante situaties de graden van buren gecorreleerd zijn door de manier waarop interacties georganiseerd zijn. Beschouw bijvoorbeeld netwerken van bedrijven die samenwerken in onderzoek en ontwikkeling. Deze netwerken bestaan uit overlappende samenwerkingsverbanden, zoals geïllustreerd in Figuur 3. Bedrijven zijn verbonden in het netwerk dan en slechts dan als ze tenminste één samenwerkingsverband gemeenschappelijk hebben.

Hoofdstuk 5 ontwikkelt een model waarin de correlatie tussen de graden van spelers voortkomt uit een dergelijke groepstructuur. We karakteriseren de typedistributie en de clustering in dit model. De clustering van een model is de conditionele kans dat twee spelers buren zijn in het netwerk, gegeven dat ze een gemeenschappelijke buur hebben. We laten zien dat we modellen voor de verwachtingen van spelers kunnen construeren met elke gewenste distributie van typen en met elke gewenste clustering. Dit is belangrijk omdat dit een mogelijkheid biedt om de invloed van clustering op strategische interacties in netwerkspelen te onderzoeken.

Deel II van dit proefschrift handelt over leerprocessen in spelen. Traditioneel houdt speltheorie zich voornamelijk bezig met de karakterisatie van evenwichten



Figuur 3. Een voorbeeld van een onderzoeksnetwerk in de biotechnologiesector. De figuur laat alle gemeenschappelijke samenwerkingsverbanden in onderzoek en ontwikkeling van zeven bedrijven (Abbott Laboratories (AL), Astra AB (AAB), Eli Lilly & Co (EL), Hoffmann-La Roche Inc. (HLR), American Home Products Corp (AHP), Millennium (M), en Pfizer (P)) in de periode 1994–2007 zien. Andere bedrijven in het netwerk zijn Genetics Institute Inc. (G), en Roche Holding AG (RH).

in spelen. De laatste jaren is er steeds meer aandacht voor leerprocessen die kunnen verklaren of en zo ja, hoe spelers leren volgens een evenwicht te spelen in een bepaald spel. Een beter begrip hiervan is noodzakelijk omdat veel strategische situaties dusdanig complex zijn dat het niet aannemelijk is dat spelers direct volgens een evenwicht van het spel zullen spelen. Bovendien kan de theorie van leerprocessen helpen om een beter inzicht te krijgen in de vraag of spelers over het algemeen volgens een bepaalde klasse evenwichten zullen spelen in plaats van volgens een andere klasse, en zo ja, waarom dat het geval is. Ook kan een studie van leerprocessen meer inzicht geven in de vraag waarom in experimenten spelers soms wel, en soms niet volgens een evenwicht leren te spelen. Tenslotte kunnen de inzichten verkregen uit de studie van leerprocessen helpen om bestaande evenwichtsconcepten te beoordelen. Zo convergeren sommige leerprocessen naar bepaalde verfijningen van Nash-evenwichten, terwijl onder andere leerprocessen de voorspellingen veel zwakker zijn dan Nash-evenwichten.

De hoofdstukken in Deel II van dit proefschrift dragen op verschillende manieren bij aan een beter inzicht in leerprocessen. Hoofdstuk 6 introduceert een leerproces waarin spelers altijd een beste antwoord spelen tegen de verwachtingen die ze hebben van de acties van hun opponenten gebaseerd op het recente verleden, maar daarbij een voorkeur hebben voor beste antwoorden die meer recent gespeeld zijn. We nemen dus aan dat, hoewel spelers rationeel zijn in de zin dat ze een beste antwoord kiezen, ze tegelijkertijd een voorkeur hebben voor meer recente acties. Dit is een realistische aanname: recent gespeelde acties liggen per definitie verser in het geheugen dan andere acties. Bovendien ontwikkelen spelers vaak gewoontes, of gebruiken ze simpele vuistregels bij het kiezen van de juiste actie, fenomenen die gemodelleerd kunnen worden door een dergelijke voorkeur voor recent gespeelde acties.

We laten zien dat dit leerproces convergeert naar zogenoemde minimale prepverzamelingen. Minimale prepverzamelingen—waarbij "prep" staat voor "preparation", oftewel voorbereiding—zijn een oplossingsconcept dat elke speler een verzameling acties voorschrijft zodanig dat hij tenminste één beste antwoord heeft tegen elke mogelijke verwachting die consistent is met de acties die voorgeschreven zijn aan zijn opponenten. Dit oplossingsconcept combineert dus de aanname dat spelers rationeel zijn (in de zin dat de verzameling voorgeschreven acties van een speler een beste antwoord moet bevatten tegen elke mogelijke verwachting die consistent is met de aan zijn opponenten voorgeschreven acties) met de aanname dat spelers bij voorkeur geen overbodige ballast met zich meetorsen, in de zin dat iedere speler een verzameling acties gebruikt die zo klein mogelijk is. Door te laten zien dat het hierboven beschreven leerproces convergeert naar minimale prepverzamelingen geven we een dynamische motivatie voor dit oplossingsconcept.

In Hoofdstuk 7 bekijken we het concept van minimale prepverzamelingen en het gerelateerde oplossingsconcept van minimale curbverzamelingen nader. Het verschil tussen minimale curbverzamelingen—curb staat voor "closed under rational behavior"—en minimale prepverzamelingen is dat minimale curbverzamelingen eisen dat de voorgeschreven actieverzameling van iedere speler *alle* beste antwoorden bevat tegen verwachtingen die consistent zijn met de voorgeschreven acties van de andere spelers, in plaats van tenminste één. Merk op dat onder beide oplossingsconcepten de oplossing van een spel niet bestaat uit een verzameling *punten* in de strategieruimte, zoals bijvoorbeeld bij het Nash-evenwicht het geval is, maar uit een verzameling *actieprofielen*. We geven een axiomatische karakterisatie van minimale prepverzamelingen en minimale curbverzamelingen. In het bijzonder laten we zien dat beide oplossingsconcepten consistent zijn in de zin dat voor beide oplossingsconcepten geldt dat, gegeven dat andere spelers volgens een bepaalde oplossing spelen, het oplossingsconcept jou voorschrijft hetzelfde te doen. Dit is een belangrijk resultaat, omdat behalve het Nash-evenwicht, geen enkel oplossingsconcept dat als oplossing van een spel een verzameling punten in de strategieruimte voorschrijft voldoet aan het axioma van consistentie (en twee andere natuurlijke axioma's waaraan minimale prepverzamelingen en minimale curbverzamelingen ook voldoen). Daarentegen voldoen minimale prepverzamelingen en minimale curbverzamelingen wel aan deze natuurlijke eigenschappen.

In Hoofdstuk 8 en 9 richten we onze aandacht op leerprocessen in een speciale klasse spelen, te weten de klasse van de minoriteitsspelen. In een minoriteitsspel moet een oneven aantal spelers kiezen tussen twee alternatieven/acties. De belangrijkste aanname is dat het voor een speler die het alternatief heeft gekozen dat door het grootste aantal spelers gekozen is loont om het andere alternatief te kiezen. Dit type spelen is een interessant model voor verschillende situaties waarin congestie een rol speelt. Deze klasse spelen is ook vanuit een theoretisch oogpunt interessant. In plaats van te coördineren op een gezamenlijke keuze, zoals in het hierboven beschreven spel waarin spelers een communicatietechnologie moeten kiezen, moeten spelers in minoriteitsspelen coördineren om te differentiëren: spelers moeten zich zo goed mogelijk "spreiden" over de twee acties. Dit lijkt, in ieder geval op het eerste gezicht, moeilijker dan te coördineren op dezelfde actie. Bovendien heeft het spel een groot aantal evenwichten. Het is dus niet direct duidelijk of en hoe spelers in deze klasse spelen volgens één van de evenwichten van het spel zullen leren te spelen.

In Hoofdstuk 8 karakteriseren we de evenwichten van deze klasse spelen, en bekijken we de voorspellingen van verschillende bekende speltheoretische leermodellen voor deze klasse spelen. We laten zien dat de voorspellingen van veelgebruikte speltheoretische leermodellen niet altijd overeenkomen. Terwijl de replicator dynamica bijvoorbeeld convergentie voorspelt naar een Nash-evenwicht met maximaal één speler die een gemengde strategie speelt, valt de verzameling stationaire punten onder de verstoorde beste-antwoord dynamica samen met de verzameling van de logit quantal-response-evenwichten van het spel. Voor minoriteitsspelen met drie spelers bestaat de verzameling Nash-evenwichten die de limiet vormt van een reeks logit quantal-response-evenwichten met afnemende ruis uit de zuivere Nash-evenwichten, de Nash-evenwichten waarin één speler beide acties met gelijke kans kiest en het Nash-evenwicht waarin alle spelers beide acties met gelijke kans kiezen.

Dat de theoretische leermodellen geen uitsluitsel kunnen geven over de vraag welk gedrag we kunnen verwachten in experimenten is in het bijzonder belangrijk omdat de uitkomsten van experimenten niet goed kunnen worden verklaard met de standaard speltheoretische leermodellen. Aan de ene kant is het geaggregeerde gedrag van experimentele subjecten in overeenstemming met de Nash-voorspelling dat spelers zich min of meer gelijk over de alternatieven verspreiden. Aan de andere kant lijken individuele subjecten niet volgens een Nash-evenwicht te spelen. Dit kan niet goed worden verklaard met standaard speltheoretische leermodellen.

We kijken daarom in Hoofdstuk 9 naar een alternatief leermodel. In dit model zijn spelers uitgerust met een stochastische verzameling vuistregels die hen voorschrijven welke actie ze moeten kiezen gegeven een geschiedenis van recente uitkomsten. Welke vuistregel spelers kiezen wordt bepaald door het succes van de verschillende vuistregels in het verleden. Doordat de verzameling vuistregels typisch zal verschillen van speler tot speler, introduceert dit heterogeneteit in het leermodel. We laten zien dat dit model onder bepaalde omstandigheden de experimentele uitkomsten goed kan verklaren: onder bepaalde condities organiseren spelers zich in verschillende groepen die steeds verschillende keuzes maken. Het kan bijvoorbeeld zo zijn dat spelers zich organiseren in twee groepen: één groep die de vuistregel gebruikt dat ze van actie wisselen wanneer de actie die zij kiezen steeds gekozen wordt door de meerderheid, en één groep met de vuistregel dat ze blijven vasthouden aan hun actie wanneer die actie steeds door de meerderheid gekozen wordt—bijvoorbeeld omdat ze verwachten dat anderen van actie zullen veranderen. Onder bepaalde omstandigheden zullen de "antagonistische" groepen ongeveer even groot zijn. Dit leidt niet tot Nash-spel op microniveau, maar wel tot een ongeveer gelijke verdeling van spelers over de twee acties, in overeenstemming met experimentele resultaten.